Architectures for Distributed Control for Performance Optimization in presence of sub-controller communication noise

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***Abstract*— In this paper, we use state space approach to give a sufficient condition for internal stability of the closed loop system when the centralized stabilizing controller is implemented in a distributive manner. Using this condition, we show that the centralized stabilizing controller for a 2-nest system can be split into two sub-controllers without affecting the internal stability. The effect of sub-controller to sub-controller communication noise on the performance is considered along with the constraint on strength of sub-controller to sub-controller communication signal. We take an input-output approach. In a 2-nest case, we obtain a sufficient condition for splitting the stabilizing controller such that the overall performance optimization can be cast as a convex problem in the Youla-Kucera parameter Q. We also present an architecture for distributive implementation of banded structure controllers such that all closed loop maps are affine in Q.**

I. Introduction

In a large complex and distributed system there is often a need of considering structure in the overall control scheme. In this paper we utilize the general framework illustrated in Figure 1 where G represents the generalized plant and K represents the controller. Both G and K are assumed to be discrete time, linear and time invariant. We will also assume that the transfer matrix G_{22} (the part of G that maps the control input u to the measured output y) is derived for a structured plant. The controller K satisfies the same structure as that of the plant.

The optimal performance problem when structural constraints are present still remains a challenge, notably the lack of a convex characterization of the problem (see for example [5], [6], [1] and references therein). Taking an input-output point of view and parameterizing all stabilizing K via the Youla-Kucera [10] parameter Q, one can see that a major source of difficulty is that structural constraints on K may lead to non-convex constraints on Q. Recently, identification of specific classes of problems for which the constraints on Q are convex with an appropriate choice of coprime factors of G was obtained in [7], [8], [4], [3]. In [2] a convex method based on solving constraints in the form of LMIs is presented to obtain structured controllers. A few architectures for distributive implementation of nested systems were first reported in [9]. In this paper, we develop an insight into designing such architectures and derive the 2-channel transmission architecture for nested system presented in [9] along with one new architecture.

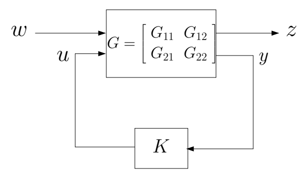

Fig. 1. The G-K interconnection

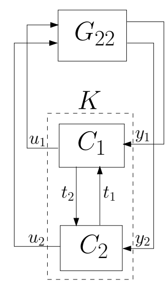

Fig. 2. Distributive implementation of K

The paper is organized as follows: in Section 2 we discuss the internal stability of systems with controllers implemented distributively. We provide conditions for internal stability of distributed systems using the state space representation. In Section 3 we consider plants with nested structure, and provide a sufficient condition to design distributed controllers respecting the nested structure such that all closed loop maps are affine in Q. We use this condition in Section 4 to obtain two architectures for distributed implementation of controllers with nested structure. An example is presented in Section 4 where we design optimal distributed controller based on these two architectures and show that they give different optimal performances. In Section 6 we consider banded structure and provide a new distributed architecture such that effect of noise on sub-controller to sub-controller signal can be characterized as an affine function of Q. We conclude in Section 7.

II. Internal Stability of distributed system

Consider distributed interconnection $G - K$ shown in Fig 2, where K is implemented distributively in terms of C_1 and C_2. The following lemmas affirm that if the induced realization of K from stabilizable and detectable state space representations of C_1 and C_2 is stabilizable and detectable then the distributed interconnection is internally stable. We assume that the distributed interconnection is well-posed.

Lemma 2.1: Consider the $G_{22} - K$ interconnection given in Fig 1 where K is a centralized stabilizing controller. The controller K is implemented distributively into two sub-controllers C_1 and C_2 as shown in Fig 2. Let, C_1 and C_2

have stabilizable and detectable state space realizations given by $\left[\begin{array}{c|c} A_{C1} & B_{C1} \\ \hline C_{C1} & 0 \end{array}\right]$ and $\left[\begin{array}{c|c} A_{C2} & B_{C2} \\ \hline C_{C2} & 0 \end{array}\right]$, respectively. If the induced realization of controller K, denoted by $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$, is stabilizable and detectable, then $G_{22} - K$ interconnection with distributed implementation is also internally stable.

Lemma 2.2: Consider $G - K$ interconnection with distributed implementation of centralized stabilizing K as shown in Fig 3, where $G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right]$ is the generalized plant with w as exogenous input and z as regulated variable. Let $\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D12 \\ C_2 & D_{21} & 0 \end{array}\right]$

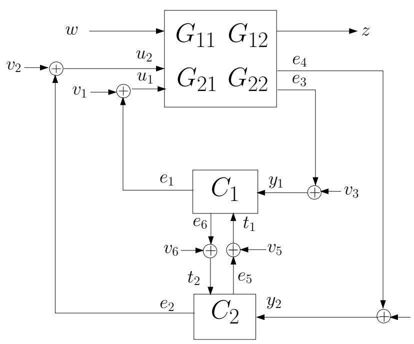

Fig. 3. $G - K$ interconnection with distributed implementation of K

be a stabilizable and detectable realization of G, with inherited realizations of G_{11}, G_{12}, G_{21} and G_{22} given by $\left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right]$, $\left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right]$, $\left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & D_{21} \end{array}\right]$ and $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$, respectively. Let, C_1 and C_2 have stabilizable and detectable state space realizations given by $\left[\begin{array}{c|c} A_{C1} & B_{C1} \\ \hline C_{C1} & 0 \end{array}\right]$ and $\left[\begin{array}{c|c} A_{C2} & B_{C2} \\ \hline C_{C2} & 0 \end{array}\right]$, respectively such that the induced realization of controller K is stabilizable and detectable. If the inherited realization of G_{22} is stablizable and detectable, then $G - K$ interconnection shown in Fig 3 is internally stable if and only if $G_{22} - K$ interconnection shown in Fig 2 is internally stable.

For nested structured controllers as shown in Fig 4, it can be shown that under certain conditions the induced realization of K from assumed stabilizable and detectable realizations of C_1 and C_2 is also stabilizable and detectable. We will use this result along with previous Lemma 2.1 and 2.2 to obtain condition for internal stability of nested interconnections.

Lemma 2.3: **(Nested System)** Consider a 2-nest $G_{22} - K$ interconnection where K is a centralized stabilizing controller implemented in distributive manner using sub-controllers C_1 and C_2 as shown in Fig 4. t is a transmitted signal from C_1 to C_2. No signal is transmitted from C_2 to C_1. Let, C_1 and C_2 have

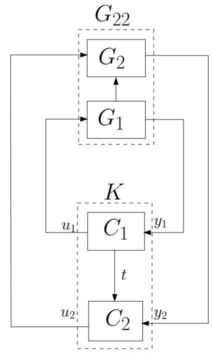

Fig. 4. 2-nest system with signal t transmitted from inner nest sub-controller C_1 to outer nest sub-controller C_2

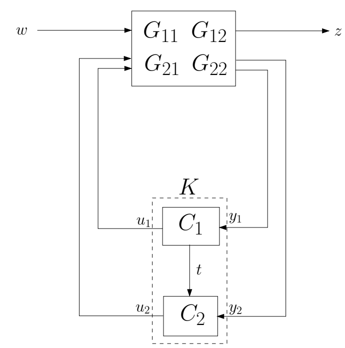

Fig. 5. 2-nest system with generalized plant G, and signal t transmitted from inner nest C_1 to outer nest C_2

state space realizations given by $\left[\begin{array}{c|c} A_{C1} & B_{C1} \\ \hline C_{C11} & \multirow{2}{*}{0} \\ C_{C12} & \end{array}\right]$ and $\left[\begin{array}{c|cc} A_{C2} & B_{C21} & B_{C22} \\ \hline C_{C2} & \multicolumn{2}{c}{0} \end{array}\right]$, respectively such that $(A_{C1}, B_{C1}, C_{C11})$ and $(A_{C2}, B_{C21}, C_{C1})$ are stabilizable and detectable. Then, the induced realization of controller K is stabilizable and detectable and $G_{22} - K$ interconnection with distributed implementation is internally stable.

Lemma 2.4: Consider the $G - K$ interconnection of a nested system shown in Fig 5, where $G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right]$ is the generalized plant with w as exogenous input and z as regulated variable. The state space realization of G is stabilizable and detectable. If the inherited state space realization of G_{22} is stabilizable and detectable, then $G-K$ interconnection shown in Fig 5 is internally stable if and only if $G_{22}-K$ interconnection in Fig 4 is internally stable.

III. Distributed Controller Design For Nested System

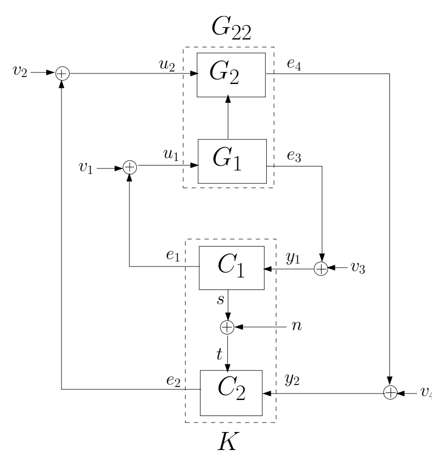

Fig. 6. 2-nest system with communication noise affecting sub-controller to sub-controller transmission

Consider a 2-nest system shown in Fig 6 with communication noise n affecting sub-controller to sub-controller transmission t. Let $G_{22} = \left[\begin{array}{c|c} G_{22a} & 0 \\ \hline G_{22c} & G_{22d} \end{array}\right]$ where $G_{22a} := u_1 \mapsto e_3, G_{22c} := u_1 \mapsto e_4$ and $G_{22d} := u_2 \mapsto e_4$. Let $C_1 = \left[\begin{array}{c} C_{1a} \\ C_{1b} \end{array}\right]$, and $C_2 = \left[\begin{array}{cc} C_{2a} & C_{2b} \end{array}\right]$ where $C_{1a} := y_1 \mapsto e_1, C_{1b} := y_1 \mapsto s, C_{2a} := t \mapsto e_2$ and $C_{2b} := y_2 \mapsto e_2$. Then $K = \left[\begin{array}{c|c} K_{11} & 0 \\ \hline K_{21} & K_{22} \end{array}\right]$, where $K_{11} = C_{1a}, K_{22} = C_{2b}$ and $K_{21} = C_{2a}C_{1b}$.

$$\left(\begin{array}{c} e_3 \\ e_4 \end{array}\right) = \left[\begin{array}{c|c} G_{22a} & 0 \\ \hline G_{22c} & G_{22d} \end{array}\right] \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right),$$
$$\left(\begin{array}{c} e_1 \\ s \end{array}\right) = \left[\begin{array}{c} C_{1a} \\ C_{1b} \end{array}\right] y_1,\ e_2 = \left[\begin{array}{cc} C_{2a} & C_{2b} \end{array}\right] \left(\begin{array}{c} t \\ y_2 \end{array}\right),$$
$$u_1 = e_1 + v_1,\ u_2 = e_2 + v_2,\ y_1 = e_3 + v_3,$$
$$y_2 = e_4 + v_4,\ t = s + n.$$

Setting $v_1 = v_2 = v_3 = v_4 = 0$, and eliminating s, t, e_1, e_2, e_3 and e_4 in above set of equations, we get the closed loop maps from noise to internal variables of interconnection as follows:

$$\Phi_{u_1 n} = 0, \Phi_{u_2 n} = (I - C_{2b}G_{22d})^{-1}C_{2a}, \Phi_{tn} = I, \quad (1)$$
$$\Phi_{y_1 n} = 0, \Phi_{y_2 n} = G_{22d}(I - C_{2b}G_{22d})^{-1}C_{2a}. \quad (2)$$

The notation used throughout this paper is that Φ_{ab} is a closed loop map from input b to output a.

Setting $n = 0$ in above set of equations, we can obtain closed loop maps from external signals v_1, v_2, v_3 and v_4 to the transmitted signal t. They are given by:

$$\Phi_{tv_1} = \left[\begin{array}{cc} C_{1b} & 0 \end{array}\right](I - G_{22}K)^{-1}\left[\begin{array}{c} G_{22a} \\ G_{22c} \end{array}\right], \quad (3)$$
$$\Phi_{tv_2} = \left[\begin{array}{cc} C_{1b} & 0 \end{array}\right](I - G_{22}K)^{-1}\left[\begin{array}{c} 0 \\ G_{22d} \end{array}\right] = 0,$$
$$\Phi_{tv_3} = \left[\begin{array}{cc} C_{1b} & 0 \end{array}\right](I - G_{22}K)^{-1}\left[\begin{array}{c} I \\ 0 \end{array}\right], \quad (4)$$
$$\Phi_{tv_4} = \left[\begin{array}{cc} C_{1b} & 0 \end{array}\right](I - G_{22}K)^{-1}\left[\begin{array}{c} 0 \\ I \end{array}\right] = 0.$$

We can parameterize all stabilizing controllers K for nested system given in Fig 4 using Youla-Kucera parameter Q. The following result translates the triangular structure restriction on the controller to the same structure on the Youla parameter Q.

Lemma 3.1: Consider 2-nest $G_{22} - K$ system shown in Fig 4, where $G_{22} = \left[\begin{array}{c|c} G_{22a} & 0 \\ \hline G_{22c} & G_{22d} \end{array}\right] := P$ maps control inputs $u = (u_1', u_2')'$ to the measured output $y = (y_1', y_2')'$. Assume that $P_1 = \left[\begin{array}{cc} G_{22a} & 0 \end{array}\right]$ and $P_2 = \left[\begin{array}{cc} G_{22c} & G_{22d} \end{array}\right]$ have state space realizations $\left[\begin{array}{c|cc} A_1 & B_{11} & 0 \\ \hline C_1 & D_{11} & 0 \end{array}\right]$ and $\left[\begin{array}{c|cc} A_2 & B_{21} & B_{22} \\ \hline C_2 & D_{21} & D_{22} \end{array}\right]$, respectively, and the inherited realizations of G_{22a} and G_{22d} are stabilizable and detectable. Then there exist stable lower triangular parameters Y_r, M_r, X_r, N_r, X_ℓ, N_ℓ, Y_ℓ, and M_ℓ satisfying the following identity

$$\left(\begin{array}{cc} X_l & -Y_l \\ -N_l & M_l \end{array}\right)\left(\begin{array}{cc} M_r & Y_r \\ N_r & X_r \end{array}\right) = I \quad (5)$$

such that the following statements are equivalent:

- K is lower triangular and it internally stabilizes the $G_{22} - K$ inter-connection.
- there exists a stable Q that is lower triangular such that

$$K = (Y_r - M_rQ)(X_r - N_rQ)^{-1} = (X_\ell - QN_\ell)^{-1}(Y_\ell - QM_\ell)$$

Proof: This is a 2−input 2−output case of generalized result on parameterization of lower triangular controller in terms of lower triangular Q parameter given in [3] ∎

This results in the parameterization of K in terms of Q having the same structure. Let Q be partitioned according to the structure of G_{22} and K:

$$Q = \left(\begin{array}{cc} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{array}\right).$$

Let, $\overline{Y}_r = Y_r - M_rQ$, $\overline{Y}_l = Y_l - QM_l$, $\overline{X}_r = X_r - N_rQ$ and $\overline{X}_l = X_l - QN_l$. Note that $\overline{Y}_r$, $\overline{Y}_l$, $\overline{X}_r$ and $\overline{X}_l$ are affine in Q, stable and have lower triangular structure. Let, $\overline{X}_l = \left(\begin{array}{cc} \widetilde{X}_{11} & 0 \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{array}\right)$; $\overline{Y}_l = \left(\begin{array}{cc} \widetilde{Y}_{11} & 0 \\ \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{array}\right)$; $N_l = \left(\begin{array}{cc} \widetilde{N}_{11} & 0 \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{array}\right)$; $M_l = \left(\begin{array}{cc} \widetilde{M}_{11} & 0 \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{array}\right)$; $\overline{X}_r = \left(\begin{array}{cc} X_{11} & 0 \\ X_{21} & X_{22} \end{array}\right)$; $\overline{Y}_r = \left(\begin{array}{cc} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{array}\right)$; $N_r = \left(\begin{array}{cc} N_{11} & 0 \\ N_{21} & N_{22} \end{array}\right)$ and $M_r = \left(\begin{array}{cc} M_{11} & 0 \\ M_{21} & M_{22} \end{array}\right)$.

Using above notations and parameterization, we can formulate a sufficient condition to design sub-controllers for the nested system that can be implemented distributively as shown in Fig 4 such that all closed loop maps are stable and affine in Q.

Lemma 3.2: Consider the 2-nest system shown in Fig 8 where $G_{22} = \left[\begin{array}{c|c} G_{22a} & 0 \\ \hline G_{22c} & G_{22d} \end{array}\right]$. Let $K(Q) = \left[\begin{array}{c|c} K_{11} & 0 \\ \hline K_{21} & K_{22} \end{array}\right]$ be a centralized stabilizing controller parameterized in terms of lower triangular Q. The controller K is implemented into two sub-controllers C_1 and C_2 where C_1 is located with inner nest and C_2 is located with outer nest and t is the transmitted signal from C_1 to C_2. Writing K in terms of sub-controllers, we have:

$$\left[\begin{array}{c|c} C_{1a} & 0 \\ \hline C_{2a}C_{1b} & C_{2b} \end{array}\right] = \left(\begin{array}{cc} \widetilde{X}_{11} & 0 \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{array}\right)^{-1}\left(\begin{array}{cc} \widetilde{Y}_{11} & 0 \\ \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{array}\right) \quad (6)$$

Then following two statements hold:

- If the interconnection with a distributive implementation of K is internally stable with all input output closed loop maps of distributed interconnection being affine in Q, then we can find sub-controllers satisfying

Equation (6) such that the distributed interconnection is internally stable and, $\widetilde{X}_{22}C_{2a}$ and $C_{1b}X_{11}$ are affine in Q.

- If the sub-controllers satisfy Equation (6) such that $\widetilde{X}_{22}C_{2a}$ and $C_{1b}X_{11}$ are stable and affine in Q, then the interconnection with distributive implementation of K is internally stable with all input output closed loop maps of distributed interconnection being affine in Q.

Proof: (a) By using fact that $(I - G_{22}K)^{-1} = \overline{X_r}M_l, (I - C_{2b}G_{22d})^{-1} = M_{22}\widetilde{X}_{22}$, $G_{22a} = \widetilde{M}_{11}^{-1}\widetilde{N}_{11}$ and $G_{22d} = \widetilde{M}_{22}^{-1}\widetilde{N}_{22}$, we can rewrite the closed loop maps given by Equation (1)-(4) as follows:

$$\Phi_{u_1n} = 0, \Phi_{u_2n} = M_{22}\widetilde{X}_{22}C_{2a}, \Phi_{y_1n} = 0, \quad (7)$$

$$\Phi_{y_2n} = N_{22}\widetilde{X}_{22}C_{2a}, \Phi_{tn} = I, \Phi_{tv_1} = C_{1b}X_{11}\widetilde{N}_{11}, \quad (8)$$

$$\Phi_{tv_2} = 0, \Phi_{tv_3} = C_{1b}X_{11}\widetilde{M}_{11}, \Phi_{tv_4} = 0. \quad (9)$$

Since, $M_{22}, N_{22}, \widetilde{N}_{11}$ and $\widetilde{M}_{11}$ are independent of Q and inverse of M_{22} and $\widetilde{M}_{11}$ exist, all closed loop maps given in Equation (7)-(9) are affine in Q implies that $\widetilde{X}_{22}C_{2a}$ and $C_{1b}X_{11}$ are affine in Q.

(b) Following the arguments of (a), if $\widetilde{X}_{22}C_{2a}$ and $C_{1b}X_{11}$ are stable and affine in Q then all closed loop maps given in Equation (7)-(9) are stable and affine in Q. ∎

Thus, the problem of splitting the centralized stabilizing controller K into two sub-controllers C_1 and C_2 such that the distributive implementation is internally stable and all closed loop maps are affine in Q is equivalent to solving Equation (6) such that $\widetilde{X}_{22}C_{2a}$ and $C_{1b}X_{11}$ affine in Q. We use this result to obtain two simple architectures for distributive implementation of K such that all closed loop maps are stable and affine in Q.

IV. Two architectures for distributed implementation of nested systems

From Lemma 3.2, we can obtain architectures for distributed implementation of K in terms of C_1 and C_2. Let, $\widetilde{X}_{22}C_{2a} := T_a$ and $C_{1b}X_{11} := T_b$, where T_a and T_b are affine in Q. Using this in $C_{2a}C_{1b} = -\widetilde{X}_{22}^{-1}\widetilde{X}_{21}\widetilde{X}_{11}^{-1}\widetilde{Y}_{11} + \widetilde{X}_{22}^{-1}\widetilde{Y}_{21}$ we get

$$T_aT_b = -\widetilde{X}_{21}Y_{11} + \widetilde{Y}_{21}X_{11}. \quad (10)$$

Note that if we transmit only one signal, then problem of finding stabilizing sub-controllers for which all closed loop maps are affine in Q is same as factorizing $(-\widetilde{X}_{21}Y_{11} + \widetilde{Y}_{21}X_{11})$ into two factors which are affine in Q. If we transmit two signals, i.e. $t = (t_1, t_2)'$, then $C_{2a} = \begin{bmatrix} C_{2a1} & C_{2a2} \end{bmatrix}$ and $C_{1b} = \begin{bmatrix} C_{1b1} \\ C_{1b2} \end{bmatrix}$. Similarly, $T_a = \begin{bmatrix} T_{a1} & T_{a2} \end{bmatrix}$ and $T_b = \begin{bmatrix} T_{b1} \\ T_{b2} \end{bmatrix}$. Thus, above equation becomes

$$T_{a1}T_{b1} + T_{a2}T_{b2} = -\widetilde{X}_{21}Y_{11} + \widetilde{Y}_{21}X_{11}. \quad (11)$$

Clearly, if we take $T_{a1} = -\widetilde{X}_{21}, T_{b1} = Y_{11}, T_{a2} = \widetilde{Y}_{21}$ and $T_{b2} = X_{11}$, we have one distributed implementation. This gives following sub-controllers:

$$C_{1a} = \widetilde{X}_{11}^{-1}\widetilde{Y}_{11}, C_{1b} = \begin{bmatrix} C_{1b1} \\ C_{1b2} \end{bmatrix} = \begin{bmatrix} \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} \\ I \end{bmatrix},$$

$$C_{2a} = \begin{bmatrix} C_{2a1} & C_{2a2} \end{bmatrix} = \begin{bmatrix} -\widetilde{X}_{22}^{-1}\widetilde{X}_{21} & \widetilde{X}_{22}^{-1}\widetilde{Y}_{21} \end{bmatrix},$$

$$C_{2b} = \widetilde{X}_{22}^{-1}\widetilde{Y}_{22}.$$

This implementation is shown in Fig 7. The closed loop transfer functions are all affine in Q and given by:

$$\Phi_{u_2n_1} = -M_{22}\widetilde{X}_{21}, \quad \Phi_{u_2n_2} = M_{22}\widetilde{Y}_{21}, \quad \Phi_{y_2n_1} = -N_{22}\widetilde{X}_{21},$$
$$\Phi_{y_2n_2} = N_{22}\widetilde{Y}_{21}, \quad \Phi_{t_1v_1} = Y_{11}\widetilde{N}_{11}, \quad \Phi_{t_2v_1} = X_{11}\widetilde{N}_{11},$$
$$\Phi_{t_1v_3} = Y_{11}\widetilde{M}_{11}, \quad \Phi_{t_2v_3} = X_{11}\widetilde{M}_{11} \quad .$$

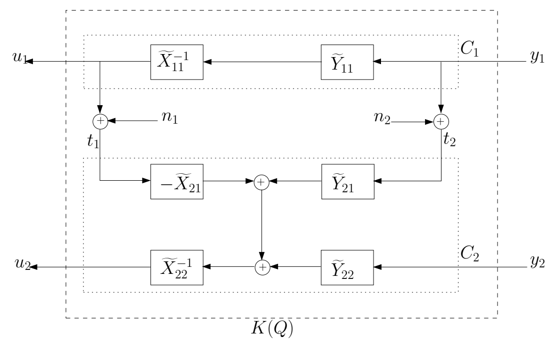

Fig. 7. Distributed implementation of K in terms of left coprime factors

This architecture in terms of left-coprime factors is same as 2-channel distributed controller architecture for nested system first presented in [9]. In this paper, we are able to obtain this architecture through the sufficient condition for distributed controller design given by Lemma 3.2.

We can further use Lemma 3.2 to obtain a new 2-channel architecture for nested system controller by noting that $-\widetilde{X}_{21}Y_{11} + \widetilde{Y}_{21}X_{11} = \widetilde{X}_{22}Y_{21} - \widetilde{Y}_{22}X_{21}$. We can rewrite equation (10) as:

$$T_{a1}T_{b1} + T_{a2}T_{b2} = -\widetilde{X}_{21}Y_{11} + \widetilde{Y}_{21}X_{11}. \quad (12)$$

Thus, if we take $T_{a1} = \widetilde{X}_{22}, T_{b1} = Y_{21}, T_{a2} = -\widetilde{Y}_{22}$ and $T_{b2} = X_{21}$, we have another distributed implementation for two channel transmission as follows:

$$C_{1a} = Y_{11}X_{11}^{-1}, C_{1b} = \begin{bmatrix} C_{1b1} \\ C_{1b2} \end{bmatrix} = \begin{bmatrix} Y_{21}X_{11}^{-1} \\ -X_{21}X_{11}^{-1} \end{bmatrix},$$

$$C_{2a} = \begin{bmatrix} C_{2a1} & C_{2a2} \end{bmatrix} = \begin{bmatrix} I & Y_{22}X_{22}^{-1} \end{bmatrix}, C_{2b} = Y_{22}X_{22}^{-1}.$$

This implementation is shown in Fig 8. The closed loop transfer functions are all affine in Q and given by:

$$\Phi_{u_2n_1} = M_{22}\widetilde{X}_{22}, \quad \Phi_{u_2n_2} = M_{22}\widetilde{Y}_{22}, \quad \Phi_{y_2n_1} = -N_{22}\widetilde{X}_{22},$$
$$\Phi_{y_2n_2} = N_{22}\widetilde{Y}_{22}, \quad \Phi_{t_1v_1} = Y_{21}\widetilde{N}_{11}, \quad \Phi_{t_2v_1} = -X_{21}\widetilde{N}_{11},$$
$$\Phi_{t_1v_3} = Y_{21}\widetilde{M}_{11}, \quad \Phi_{t_2v_3} = -X_{21}\widetilde{M}_{11} \quad .$$

These two architectures using 2 channels for transmission

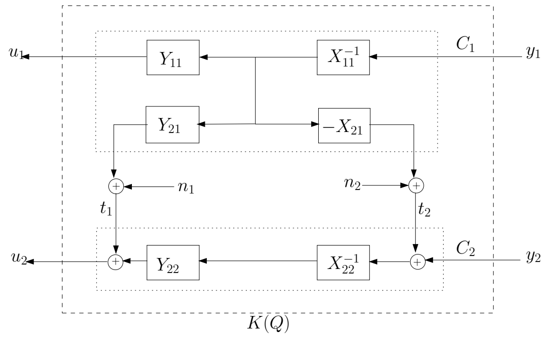

Fig. 8. Distributed implementation of K in terms of right coprime factors

between sub-controllers have all closed loop maps affine in Q which leads to a convex performance optimization problem for finding optimal controller for each architecture. These results for 2-nest system can be generalized to n-nest system. Depending on the given nested system and performance measures, one architecture can be better than the other. Thus, it matters what architecture we choose to meet system performance requirement as demonstrated by the example presented in the next section.

V. Example: Optimal distributed control design for 2-node ABR network ni presence of noise

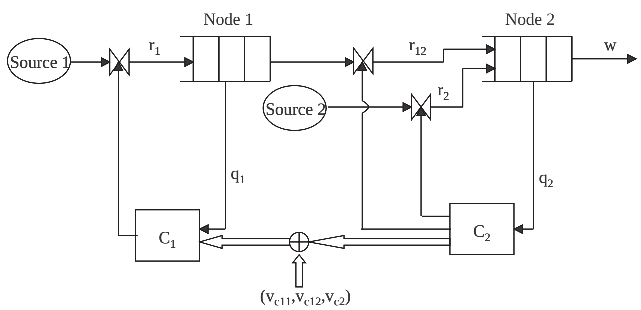

Fig. 9. 2-nodal ABR network with congestion control

In this example, we consider a 2-node ABR communication network as shown in Fig 9. We will show that the two architectures obtained in previous section will give different optimal controllers and different values of performance measure. The objective is to design a distributed controller for given ABR network which not only avoids the congestion in the network while keeping the channel utilization ratio as large as possible, but also minimizes the effect of sub-controller to sub-controller noise on queue lengths (q_1 and q_2) and regulated rates (r_1, r_{12} and r_2) of transmission of packets by regulating rates r_1 and r_2. We are also interested in minimizing the signal power of transmitted signal between sub-controllers. w is the total available capacity (bit-rate) for the two sources. The overall controller K consists of two sub-controllers C_1 and C_2, controlling r_1 and (r_{12}, r_2), respectively. The information transmitted from C_2 to C_1 are (r_{12}, r_2) and q_2, which get corrupted by communication noise viz. v_{c11}, v_{c12} and v_{c2}, respectively. The regulated variable is $z = \{(r_2 - a_1 w), (r_1 - a_2 w), q_2, q_1\}^T$, where a_1=a_2=0.5 so that each of the nodes gets half of the available capacity w. Further, we assume that w is typically step signal to be tracked.

Thus, we want to find a stabilizing controller K which is lower triangular and minimizes $\|T(K)\|$, where $T(K)$: $\begin{pmatrix} w \\ n \end{pmatrix} \mapsto \begin{pmatrix} z \\ t \end{pmatrix} = \begin{bmatrix} \Phi_{zw} & \Phi_{zn} \\ \Phi_{tw} & \Phi_{tn} \end{bmatrix}$, where Φ_{zw} captures the performance requirement of system, Φ_{zn} captures the effect of sub-controller communication noise on performance and Φ_{tw} denotes the power of transmitted signal with respect to power of external input signals. From Lemma 3.1, we can write $T(K) = T(Q)$, where Q is lower triangular and stable. This parametrization makes the closed loop map Φ_{zw} affine in Q, i.e. $\Phi_{zw} = H - U * Q * V$, where $H = G_{11} + G_{12} Y_r M_l G_{21}, U = G_{12} M$ and $V = M_l G_{21}$. $\Phi_{tn} = I$ (from Equation (8)), while Φ_{zn} and Φ_{tw} depend on sub-controllers C_{1b} and C_{2a}.

$$\Phi_{zn} = G_{12} \begin{bmatrix} \Phi_{u_1,n} \\ \Phi_{u_2,n} \end{bmatrix} =: \begin{bmatrix} G_{12}^{u_1} & G_{12}^{u_2} \end{bmatrix} \begin{bmatrix} 0 \\ M_{22} \widetilde{X}_{22} C_{2a} \end{bmatrix}$$

$$\Phi_{tw} = \begin{bmatrix} \Phi_{t,v_3} & \Phi_{t,v_4} \end{bmatrix} G_{21} =: \begin{bmatrix} C_{1b} X_{11} \widetilde{M}_{11} & 0 \end{bmatrix} \begin{bmatrix} G_{21}^{v_3} \\ G_{21}^{v_4} \end{bmatrix}$$

From Section IV, we have two possible architectures, the first architecture based on left coprime factors as shown in Fig 7 and the second based on right coprime factors as shown in Fig 8, which give sub-controllers such that the closed loop maps are stable and affine in Q. Let, $T_L(K)$ and $T_R(K)$ be closed loop maps T when C_{2a} and C_{1b} are given by the first and the second architecture, respectively. Thus, the performance optimization problem can be written as follows:

$$I : \mu_L = \underbrace{\inf}_{\substack{K\text{- stabilizing, lower} \\ \text{triangular, based on} \\ \text{the first architecture}}} \|T_L(K)\|_1 = \underbrace{\inf}_{\substack{Q \text{ - stable,} \\ \text{lower} \\ \text{triangular}}} \|T_L(Q)\|_1$$

$$II : \mu_R = \underbrace{\inf}_{\substack{K\text{- stabilizing, lower} \\ \text{triangular, based on} \\ \text{the second architecture}}} \|T_R(K)\|_1 = \underbrace{\inf}_{\substack{Q \text{ - stable,} \\ \text{lower} \\ \text{triangular}}} \|T_R(Q)\|_1$$

Solving above two optimization problems for these two architectures, we obtain two different optimal controllers $Q_{opt,L}$ and $Q_{opt,R}$ with $\mu_L = 1.5$ and $\mu_R = 3.7$, respectively. Clearly, in this example it is better to use the second architecture with $\mu_L = 1.5$.

VI. Banded Structure

In this section, we consider banded structures for the controller K and provide a distributed architecture for its implementation such that the interconnection is internally stable in presence of sub-controller communication noise and all closed loop maps are affine in Q. The banded system structure is characterized by one step delay in subsystem interactions. From [3] we have that all banded controllers can be appropriately parameterized in terms of the Youla-Kucera parameter Q which has same banded structure. We use this parameterization, and present an architecture for which all closed loop maps in presence of sub-controller to sub-controller communication noise are stable and affine in Q. For simplicity, we will consider $n = 2$ case, and result can be easily generalized to any n case.

Proposition 6.1: (**Inter-connection using** 2 **channel for transmission in one direction**) Suppose the banded $G_{22} - K$ inter-connection is stable in absence of any sub-controller to sub-controller communication noise (i.e. in Fig 10 the closed loop map $(v_1, v_2, v_3, v_4)^T \mapsto (u_1, u_2, y_1, y_2)^T$ is stable). Then, in the realization of $G_{22} - K$ interconnection in Fig 10, the closed loop map $(v_1, v_2, v_3, v_4, v_{c1}, v_{c2}, v_{c3}, v_{c4})^T \mapsto (u_1, u_2, y_1, y_2, s_{11}, s_{12}, s_{21}, s_{22})^T$ is internally stable and affine in Q.

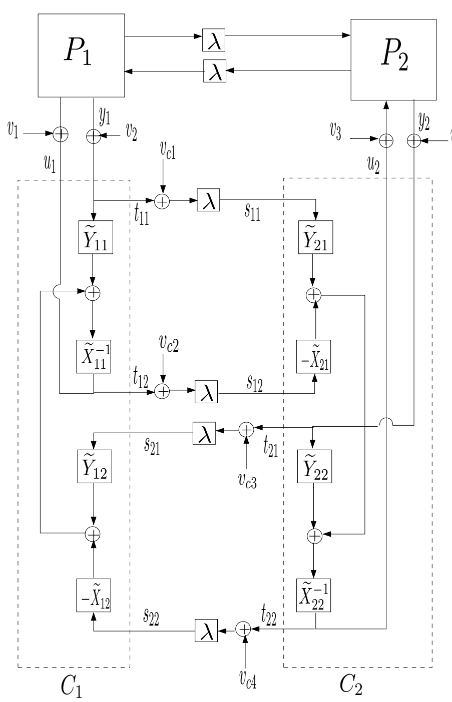

Fig. 10. Realization of banded controller using two channels for transmission in each direction.

Proof : The banded $G_{22} - K$ inter-connection is stable in absence of any sub-controller to sub-controller communication noise. Note that when $v_{c1} = 0$, $v_{c2} = 0$, $v_{c3} = 0$, $v_{c4} = 0$, we have $s_{11} = \lambda y_1$, $s_{12} = \lambda u_1$, $s_{21} = \lambda y_2$ and $s_{22} = \lambda u_2$. Thus, the closed loop map from $(v_1, v_2, v_3, v_4)^T \mapsto (u_1, u_2, y_1, y_2, s_{11}, s_{12}, s_{21}, s_{22})^T$ is internally stable and affine in Q which has banded structure [3]. Now, we must show that the closed loop map $(v_{c1}, v_{c2}, v_{c3}, v_{c4})^T \mapsto (u_1, u_2, y_1, y_2, s_{11}, s_{12}, s_{21}, s_{22})^T$ is internally stable and all the closed loop maps are affine in Q. We can show that map from $(v_{c1}, v_{c2}, v_{c3}, v_{c4})^T := v$ to $(u_1, u_2, y_1, y_2)^T$ is given by:

$$\begin{bmatrix} -\lambda^2 M_{12}\widetilde{X}_{12} & \lambda^2 M_{12}\widetilde{Y}_{12} & -\lambda M_{11}\widetilde{X}_{21} & \lambda M_{11}\widetilde{Y}_{21} \\ -\lambda^2 M_{22}\widetilde{X}_{12} & \lambda^2 M_{22}\widetilde{Y}_{12} & -\lambda M_{21}\widetilde{X}_{21} & \lambda M_{21}\widetilde{Y}_{21} \\ -\lambda^2 N_{12}\widetilde{X}_{12} & \lambda^2 N_{12}\widetilde{Y}_{12} & -\lambda N_{11}\widetilde{X}_{21} & \lambda N_{11}\widetilde{Y}_{21} \\ -\lambda^2 N_{22}\widetilde{X}_{12} & \lambda^2 N_{22}\widetilde{Y}_{12} & -\lambda N_{21}\widetilde{X}_{21} & \lambda N_{21}\widetilde{Y}_{21} \end{bmatrix}.$$

Since, all these factors are stable and M and N does not depend on Q, it implies that these maps are stable and affine in Q. Set $v_1 = 0$, $v_2 = 0$, $v_3 = 0$ and $v_4 = 0$. Then, we have $s_{11} = \lambda(y_1 + v_{c1})$, $s_{12} = \lambda(u_1 + v_{c2})$, $s_{21} = \lambda(y_2 + v_{c3})$ and $s_{22} = \lambda(u_2 + v_{c4})$. And, map from $(v_{c1}, v_{c2}, v_{c3}, v_{c4})^T$ to $(u_1, u_2, y_1, y_2)^T$ is stable and affine in Q. Thus, map from $(v_{c1}, v_{c2}, v_{c3}, v_{c4})^T$ to $(s_{11}, s_{12}, s_{21}, s_{22})^T$ is stable and affine in Q. ■

VII. Conclusion

In this paper, a sufficient condition for the internal stability of a distributed control system is given. This framework is used to derive two architectures for implementation of stabilizing controllers with nested structure such that all closed loop maps are affine in Q. The two architectures can give different performance depending on the plant and performance requirements. We have also presented a new architecture that respects the banded structure of the stabilizing controller when sub-controller to sub-controller noise is included.

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