Stabilization of nested systems with uncertain subsystem communication channels

Vikas Yadav
Dept. of Electrical and Computer Engg.,
Iowa State University,
Ames, Iowa, USA

Petros G. Voulgaris
Dept. of Aeronautical and Astronomical Engg.,
Coordinated Science Lab,
University of Illinois at Urbana-Champaign,
Urbana, IL, USA

Murti V. Salapaka
Dept. of Electrical and Computer Engg.,
Iowa State University,
Ames, Iowa, USA

***Abstract*— This paper addresses the design of stabilizing controllers for a nested control system where the controller is realized in a distributed manner, considering uncertainty not only in the controller-to-plant and plant-to-controller channels but also in the nest-to-nest, subcontroller-to-subcontroller communication. An input-output approach is taken. Two appropriate controller architectures that stabilize the entire system with the various components subject to uncertainty and noise are addressed. We present controller synthesis procedures to address deterministic uncertainty and stochastic uncertainty, that model packet-loss in the Internet.**

I. INTRODUCTION

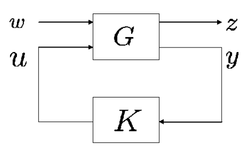

Fig. 1. $G-K$ framework.

In large complex and distributed systems there is often a need of considering structure in the overall control scheme. In this paper we will utilize the general framework illustrated in Figure 1 where G represents the generalized plant and K represents the controller. Both G and K are assumed to be discrete time, linear and time invariant. The controller K has to satisfy the nested structure that is characterized by a block triangular structure of the transfer matrix K. In addition we will also assume that the transfer matrix G_{22} (the part of G that maps the control input u to the measured output y) is derived for a nested plant and thus is also a block triangular matrix.

The optimal performance problem when structural constraints are present still remains a challenge, notably the lack of a convex characterization of the problem (see for example [8], [9], [2] and references therein). Taking an input-output point of view and parameterizing all stabilizing K via the Youla-Kucera [12] parameter Q, one can see as a major source of difficulty is that structural constraints on K may lead to non-convex constraints on Q. Recently identification of specific classes of problems for which the constraints on Q are convex with the appropriate choice of the coprime factors of G was obtained [10], [11], [7].

In this paper we address the problem of synthesizing controllers such that an appropriate part of the controller is realized at a compatible part of the nested structure. This scenario corresponds to the case where the entire controller is not realized at one central location but at possibly many different sites. This problem is motivated partly because of the upcoming need of controlling vast numbers of coupled plants that might arise for example in the control of massively parallel micro-cantilevers. It is often not feasible to provide resources at one central location to handle such tasks and therefore the control has to be achieved in a distributed manner. This issue raises the important question on how to divide the control task into the various stations and evaluate and address the noise and uncertainty that affects such an architecture. Another aspect that is pursued is when the sub-components of the controller interact with one another via the Internet. We pursue these issues in an input-output framework where we first obtain appropriate coprime factors of the plant that facilitate posing the triangular structure constraints of the controller (that appear on the closed loop maps in a nonconvex manner) as convex constraints when translated to the Youla parameter Q. This establishes that triangular structure can be effectively inherited by the Youla parameter Q. The problem of noise affecting the various components of the controller is addressed next where different architectures are presented. The issue of stability of the entire interconnection is evaluated. It is observed that certain implementation schemes provide an affine parameterization of the input-output maps in terms of Youla parameter Q with a need to transmit more than one signal between the various sub-components of the controller implementation. The issue of uncertainty in the channels used to transmit the signals between the controller sub-components is addressed by casting the problem into the standard $G-K-\Delta$ robust control framework. The uncertainty characterization are obtained in the standard ℓ_1 setting and also for the case when communication between different components of the interconnection occurs via the Internet using the erasure channel model[1]. Solution methodologies for the resulting robust synthesis problem are indicated. These solution methodologies provide for global solutions even though the underlying problems are non-convex.

The paper is organized as follows: in Section 2 we provide architectures that guarantee internal stability in the presence of sub-controller to sub-controller noise. It is shown that when two channels are utilized an affine parameterization of all closed loop maps achievable via stabilizing controllers that respect the nested structure can be obtained. An example is presented in Section 3. In Section 4 we study the associated analysis and synthesis problems that arise when uncertainty is included in the nest to nest, sub-controller to sub-controller

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communication, including the erasure uncertainty model[1]. We conclude in Section 5.

II. Youla-Kucera Parameterization of K

In this paper, we consider a nested control system, where both G_{22} and K have lower triangular structures as shown in Figure (2). Though the problem of focus in this paper can be generalized for n-nest systems, we consider only two nests for convenience of presentation. The inner nest (subsystem) consists of a plant G_1 and a controller K_1, and the outer nest (subsystem) consists of a plant G_2 and a controller K_2. The controllers K_1 and K_2 are connected through a channel so that information can be transmitted from K_1 to K_2. Similarly, nominal plants G_1 and G_2 are connected through a channel so that the control action on G_1 affects G_2 but not vice versa. The inner and outer subsystems have control inputs u_1 and u_2 and measured outputs y_1 and y_2, respectively. G_{22} has a lower triangular structure and the same structure is imposed on the controller K. Thus, G_{22} and K can be written as $G_{22} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{pmatrix}$, $K = \begin{pmatrix} K_{11} & 0 \\ K_{12} & K_{22} \end{pmatrix}$. The following result translate triangular structure of the controller K to the same structure on the Youla parameter Q.

Lemma 2.1: [6] Let the plant $P = G_{22}$ that maps the control inputs $u = (u_1, \ldots, u_n)^T$ to the measured outputs $y = (y_1, \ldots, y_n)^T$ be lower triangular. Assume that the i^{th} row of P denoted by $P_i = G_i = \{P_{i1} \ \ldots \ P_{ii} \ 0 \ \ldots, 0\}$ admits a realization $\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right]$ with $B_i = \{B_{i1} \ \ldots \ B_{ii} \ 0 \ \ldots 0\}$, $D_i = (D_{i1} \ D_{i2} \ \ldots D_{ii} \ 0 \ \ldots 0)$ and $\left[\begin{array}{c|c} A_i & B_{ii} \\ \hline C_i & D_{ii} \end{array}\right]$ the inherited state space realization of P_{ii} is stabilizable and detectable. Then there exist stable lower triangular parameters $Y_r,\ M_r,\ X_r,\ N_r,\ X_\ell,\ N_\ell,\ Y_\ell,$ and M_ℓ satisfying the following identity

$$\begin{pmatrix} X_l & -Y_l \\ -N_l & M_l \end{pmatrix} \begin{pmatrix} M_r & Y_r \\ N_r & X_r \end{pmatrix} = I \qquad (1)$$

such that the following statements are equivalent:

- K is lower triangular and it internally stabilizes the inter-connection depicted in Figure (2).
- there exists a stable Q that is lower triangular such that

$$\begin{aligned} K &= (Y_r - M_r Q)(X_r - N_r Q)^{-1} \\ &= (X_\ell - Q N_\ell)^{-1}(Y_\ell - Q M_\ell) \end{aligned}$$

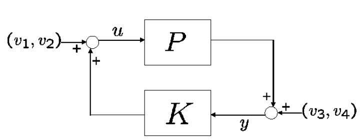

Fig. 2. Feedback inter-connection of $P = G_{22}$ and K.

This results in the parameterization of K in terms of Q having the same structure. Let Q be partitioned according to the structure of G_{22} and K as $Q = \begin{pmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{pmatrix}$.

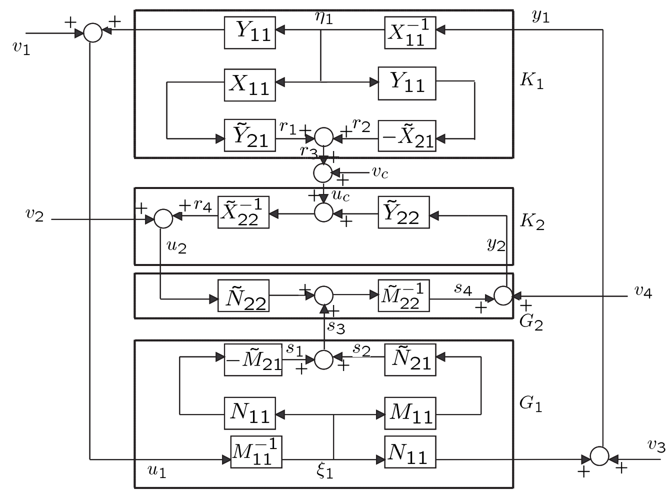

Fig. 3. Realization of $G_{22} - K$ the inter-connection with single channel transmission with noise.

Proposition 2.1: (**Stability of the Inter-connection using only one channel for transmission**) Suppose the nested $G_{22} - K$ inter-connection as shown in Figure (2) is stable with assumptions and notation of Lemma 2.1 (i.e. the closed loop map $(v_1, v_2, v_3, v_4)^T \mapsto (u_1, u_2, y_1, y_2)^T$ is stable). Then, in the realization of $G_{22} - K$ interconnection in Figure (3), the closed loop map $(v_1, v_2, v_3, v_4, v_c)^T \mapsto (u_1, u_2, y_1, y_2, u_c)^T$ is internally stable.

Proof :Assuming the notation and results of Lemma 2.1, let, $\overline{Y}_r = Y_r - M_r Q$, $\overline{Y}_l = Y_l - Q M_l$, $\overline{X}_r = X_r - N_r Q$ and $\overline{X}_l = X_l - Q N_l$. Note that $\overline{Y}_r$, $\overline{Y}_l$, $\overline{X}_r$ and $\overline{X}_l$ are affine in Q, stable and have lower triangular structure. Also let, $\overline{X}_l = \begin{pmatrix} \widetilde{X}_{11} & 0 \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{pmatrix}$; $\overline{Y}_l = \begin{pmatrix} \widetilde{Y}_{11} & 0 \\ \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{pmatrix}$; $N_l = \begin{pmatrix} \widetilde{N}_{11} & 0 \\ \widetilde{N}_{21} & \widetilde{N}_{22} \end{pmatrix}$; $M_l = \begin{pmatrix} \widetilde{M}_{11} & 0 \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{pmatrix}$; $\overline{X}_r = \begin{pmatrix} X_{11} & 0 \\ X_{21} & X_{22} \end{pmatrix}$; $\overline{Y}_r = \begin{pmatrix} Y_{11} & 0 \\ Y_{21} & Y_{22} \end{pmatrix}$; $N_r = \begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}$ and $M_r = \begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}$. Thus, from Equation (1) we get $\begin{pmatrix} \overline{X}_l & -\overline{Y}_l \\ -N_l & M_l \end{pmatrix} \begin{pmatrix} M_r & \overline{Y}_r \\ N_r & \overline{X}_r \end{pmatrix} = I$. Moreover, $K = \begin{pmatrix} \widetilde{X}_{11}^{-1}\widetilde{Y}_{11} & 0 \\ -\widetilde{X}_{22}^{-1}\widetilde{X}_{21}\widetilde{X}_{11}^{-1}\widetilde{Y}_{11} + \widetilde{X}_{22}^{-1}\widetilde{Y}_{21} & \widetilde{X}_{22}^{-1}\widetilde{Y}_{22} \end{pmatrix}$ $= \begin{pmatrix} Y_{11}X_{11}^{-1} & 0 \\ Y_{21}X_{11}^{-1} - Y_{22}X_{22}^{-1}X_{21}X_{11}^{-1} & Y_{22}X_{22}^{-1} \end{pmatrix}$.

First we will establish that the map $(v_1, v_2, v_3, v_4) \mapsto (u_1, u_2, y_1, y_2, u_c)$ is stable. Setting $v_c = 0$, we have $u_1 = K_{11}y_1 + v_1 = \widetilde{X}_{11}^{-1}\widetilde{Y}_{11}y_1 + v_1 = Y_{11}X_{11}^{-1}y_1 + v_1$ and $u_2 = K_{21}y_1 + K_{22}y_2 + v_2 = \widetilde{X}_{22}^{-1}(-\widetilde{X}_{21}Y_{11} + \widetilde{Y}_{21}X_{11})X_{11}^{-1}y_1 + \widetilde{X}_{22}^{-1}\widetilde{Y}_{22}y_2 + v_2$. This is realized by the implementation scheme shown in Figure (3) with $v_c = 0$. Since, $G_{22} - K$ inter-connection in Figure (2) is internally stable, the map in Figure (3) from $(v_1, v_2, v_3, v_4)^T \mapsto (u_1, u_2, y_1, y_2)^T$ is stable and affine in Q. From Figure (3) we obtain $u_c = r_3 = r_1 +$

$r_2 = (-\widetilde{X}_{21}Y_{11} + \widetilde{Y}_{21}X_{11})X_{11}^{-1}y_1 = -\widetilde{X}_{21}u_1 + \widetilde{Y}_{21}y_1$. As $\widetilde{X}_{21}$ and $\widetilde{Y}_{21}$ are stable factors, the map from $(v_1, v_2, v_3, v_4)^T \mapsto u_c$ is stable and therefore the map $(v_1, v_2, v_3, v_4) \mapsto (u_1, u_2, y_1, y_2, u_c)$ is stable, but not affine in Q.

Now we establish that the map $v_c \mapsto (u_1, u_2, y_1, y_2, u_c)$ is stable. We set $v_1 = v_2 = v_3 = v_4 = 0$. Then the closed loop map from $v_c \mapsto (u_1, y_1)^T$ is 0, as v_c does not affect the outer loop and the closed loop map from $v_c \mapsto u_c$ is identity and thus stable. Thus, we must show that the closed loop map $v_c \mapsto (u_2, y_2)^T$ is stable. Note that $u_c + \widetilde{Y}_{22}y_2 = \widetilde{X}_{22}u_2$ and $s_3 + \widetilde{N}_{22}u_2 = \widetilde{M}_{22}y_2$.

$$\Rightarrow \begin{pmatrix} u_c \\ s_3 \end{pmatrix} = \begin{pmatrix} \widetilde{X}_{22} & -\widetilde{Y}_{22} \\ -\widetilde{N}_{22} & \widetilde{M}_{22} \end{pmatrix} \begin{pmatrix} u_2 \\ y_2 \end{pmatrix}. \qquad (2)$$

Noting that $\begin{pmatrix} \widetilde{X}_{22} & -\widetilde{Y}_{22} \\ -\widetilde{N}_{22} & \widetilde{M}_{22} \end{pmatrix}^{-1} = \begin{pmatrix} M_{22} & Y_{22} \\ N_{22} & X_{22} \end{pmatrix}$, we get $\begin{pmatrix} u_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} M_{22} & Y_{22} \\ N_{22} & X_{22} \end{pmatrix} \begin{pmatrix} u_c \\ s_3 \end{pmatrix}$. Since, $u_c = v_c + r_3$, the closed loop map $v_c \mapsto (u_2, y_2)^T$ is given by $\begin{pmatrix} u_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} M_{22} \\ N_{22} \end{pmatrix} v_c$. This map is stable because M_{22} and N_{22} are stable factors. This proves the proposition. ∎

Figure (3) provides an architecture to implement the controller in a distributed manner where the components K_1 and K_2 are located at different sites, and only one channel is used to transmit signal from K_1 to K_2. Next we use two channels and obtain affine parameterization in terms of Youla Parameter Q of all involved close loop maps.

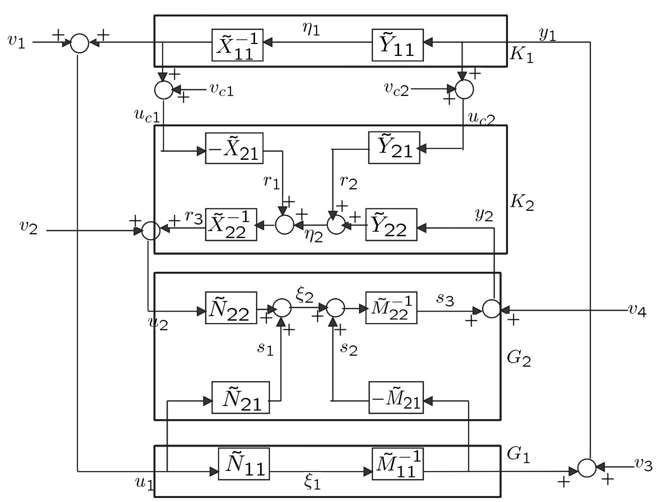

Fig. 4. Realization of the inter-connection with two channel transmission with noise.

Proposition 2.2: (**Stability of the Inter-connection using two channels for transmission**) Suppose the nested $G_{22} - K$ inter-connection as shown in Figure (2) is stable with assumptions and notations of Lemma 2.1 (i.e. the closed loop map $(v_1, v_2, v_3, v_4)^T \mapsto (u_1, u_2, y_1, y_2)^T$ is stable). Then, the inter-connection shown in Figure (4) is internally stable. Also, all achievable closed loop maps, $(v_1, v_2, v_3, v_4, v_{c1}, v_{c2}) \mapsto (u_1, u_2, y_1, y_2, u_{c1}, u_{c2})$, are affine in Q.

Proof: First we establish that the map $(v_1, v_2, v_3, v_4) \mapsto (u_1, u_2, y_1, y_2, u_{c1}, u_{c2})$ is stable and affine in the parameter Q. Setting v_{c1}=v_{c2}=0 we have u_1=$K_{11}y_1 + v_1$=$\widetilde{X}_{11}^{-1}\widetilde{Y}_{11}y_1 + v_1$ and u_2=$K_{21}y_1 + K_{22}y_2 + v_2$=$-\widetilde{X}_{22}^{-1}\widetilde{X}_{21}(u_1 - v_1) + \widetilde{X}_{22}^{-1}\widetilde{Y}_{21}y_1 + \widetilde{X}_{22}^{-1}\widetilde{Y}_{22}y_2 + v_2$. See Figure (4). Since, $G_{22} - K$ inter-connection is internally stable, the map $(v_1, v_2, v_3, v_4)^T \mapsto (u_1, u_2, y_1, y_2)^T$ is stable and affine in Q. It is evident that $(u_{c1}, u_{c2})^T = (u_1, y_1)^T$, (when $v_{c1} = v_{c2} = 0$). This proves that $(v_1, v_2, v_3, v_4) \mapsto (u_1, u_2, y_1, y_2, u_{c1}, u_{c2})$ is stable and affine in the parameter Q. Now we prove that the closed loop map $(v_{c1}, v_{c2})^T \mapsto (u_1, u_2, y_1, y_2, u_{c1}, u_{c2})^T$ is stable and affine in Q. Set $v_1 = v_2 = v_3 = v_4 = 0$. The closed loop map $(v_{c1}, v_{c2})^T \mapsto (u_1, y_1)^T$ is 0, as there is no flow of signal from the inside to outside, the closed loop maps $v_{c1} \mapsto u_{c1}$ and $v_{c2} \mapsto u_{c2}$ are identity, and $v_{c1} \mapsto u_{c2}$ and $v_{c2} \mapsto u_{c1}$ are 0. From Figure (4), we have $r_1 + r_2 + \widetilde{Y}_{22}y_2$=$\widetilde{X}_{22}u_2$ and $s_1 + s_2 + \widetilde{N}_{22}u_2$=$\widetilde{M}_{22}y_2$. Noting that $r_1 = -\widetilde{X}_{21}v_{c1}$, and $r_2 = \widetilde{Y}_{21}v_{c2}$, we obtain the closed loop map $(v_{c1}, v_{c2})^T \mapsto (u_2, y_2)^T$

$$\begin{pmatrix} u_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -M_{22}\widetilde{X}_{21} \\ -N_{22}\widetilde{X}_{21} \end{pmatrix} v_{c1} + \begin{pmatrix} M_{22}\widetilde{Y}_{21} \\ N_{22}\widetilde{Y}_{21} \end{pmatrix} v_{c2}. \qquad (3)$$

This map is stable because M_{22}, $\widetilde{X}_{21}$, N_{22}, $\widetilde{Y}_{21}$ are stable factors. It is affine in Q because $\widetilde{X}_{21}$ and $\widetilde{Y}_{21}$ are affine in Q while M_{22} and N_{22} do not depend on Q. This proves the proposition. ∎

The above result for two-nest inter-connection can be generalized to n-nest inter-connection.

III. Example: Optimal Control Design for 2-node ABR Network in presence of noise

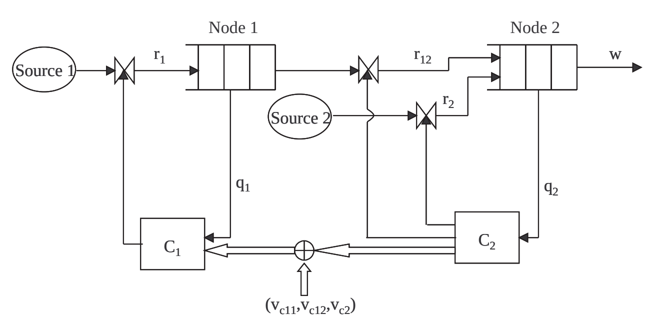

Fig. 5. 2-nodal ABR network with congestion control

The purpose of this example to illustrate the design of optimal control using the architecture of Proposition 2.2, which not only avoids the congestion in above 2-node ABR network while keeping the channel utilization ratio as large as possible, but also minimizes the affect of the noise on the queue lengths (q_1 and q_2) and the regulated rates (r_1, r_{12} and r_2) of transmission of packets by regulating rates r_1 and r_2. See Figure (5). w is the total available capacity (bit-rate) for the two sources. The overall controller consists of two sub-controllers C_1 and C_2, controlling r_1 and (r_{12}, r_2), respectively. The objectives are to avoid queue overflow, to maximize the network utilization ratio, i.e. to make $r_1 + r_2$ match w as close as possible, and to minimize the affect of noise. From Proposition 2.2, if we implement controller K as shown in Figure (4), then K internally stabilizes the network in the presence of communication uncertainty between C_1 and C_2 modeled as noise. The information transmitted from C_2 to C_1 are (r_{12}, r_2) and q_2, and they get corrupted by noise viz. v_{c11}, v_{c12} and v_{c2}, respectively. The regulated variable is $z = \{(r_2 - a_1 w), (r_1 - a_2 w), q_2, q_1\}^T$, where a_1=a_2=0.5 so that each of the nodes gets half of the available capacity w. Further, we assume that w is typically step signal to be tracked. The dynamics of the network is given by:

- Node1: $q_1(k+1) = q_1(k) + r_1(k) - r_{12}(k)$
- C_1: $r_1 = f_1(q_1, r_2 + v_{c11}, r_{12} + v_{c12}, q_2 + v_{c2})$
- Node2: $q_2(k+1) = q_2(k) + r_2(k) + r_{12}(k) - w(k)$
- C_2: $r_2 = f_2(q_2); r_{12} = f_{12}(q_2)$

where f_1, f_2 and f_{12} are causal linear operators. Clearly, plant G_{22} and controller K in this case are lower triangular operators. Let, $R_1(K)$ be the closed loop map w to z and $R_2(K)$ is the closed loop map $(v_{c11}, v_{c12}, v_{c2})^T$ to z. The overall objective is to find the lower triangular stabilizing controller which stabilizes $R_1(K)$ and minimize the sum of ℓ_1-norm of $R_1(K)$ and square of H_2-norm of $R_2(K)$. In this case, we find such stabilizing controller by solving following weighted optimization problem:

$$\mu := \inf c\|R_1(Q)\|_1 + (1-c)\|R_2(Q)\|_2^2$$

subject to: Q is stable and Q is lower triangular.

This problem is solved using GMO 1.0 ([5]), and the optimal Q (and K), and the values of $\|R_1(Q)\|_1$ and $\|R_2(Q)\|_2^2$ are obtained for c = 0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1. The Pareto trade-off curve between two optimums is plotted in Figure (6), which also gives the desired controller for a special case of $c_1 = c_2$. The minimum norms obtained in this case are $\|R_1(Q)\|_1$ = 1.34, and $\|R_2(Q)\|_2^2$ = 2.33. The square of H_2-norm is less than that obtained by minimizing the ℓ_1 and H_2 norm of R_1 only (in that case, minimum value for $\|R_2(Q)\|_2^2$ = 6.58). The optimal K of order 3, obtained in former case is

$$K_{opt}(\lambda) = \left[\begin{array}{c|c} \frac{-0.7852+0.1522\lambda-0.05008\lambda^2}{1+0.0034\lambda+0.8458\lambda^2} & 0 \\ \frac{-0.01143-0.08101\lambda-0.0345\lambda^2}{1+0.0034\lambda+0.8458\lambda^2} & 0 \\ \hline \frac{-0.3103-0.06125\lambda-0.06179\lambda^2-0.03162\lambda^3}{1+0.8034\lambda+0.08731\lambda^2-0.06767\lambda^3} & \frac{-0.81\lambda}{1+0.8\lambda} \end{array}\right].$$

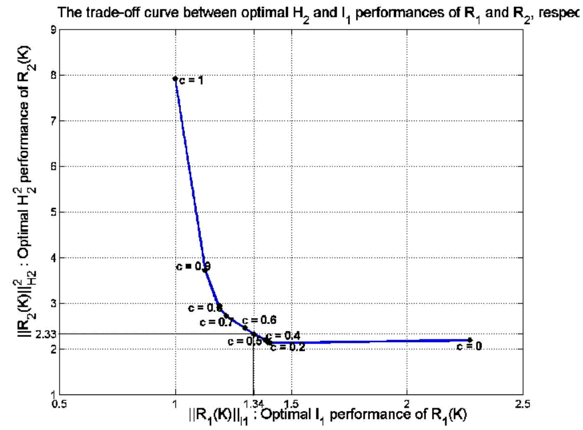

Fig. 6. The trade-off curve between the ℓ_1 and H_2 optimal performances of R_1 and R_2. respectively.

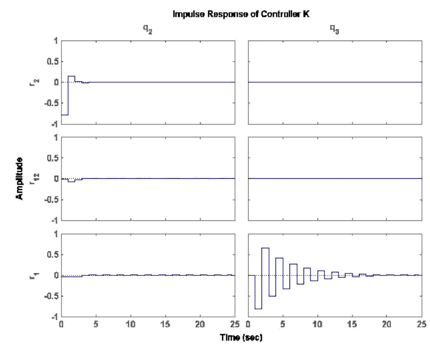

Fig. 7. Impulse Response of desired Decentralized, Lower-triangular Controller

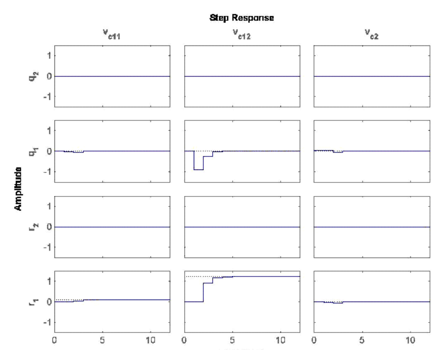

Fig. 8. Step Response of Closed-loop System From Noise to Regulated Variable z with Decentralized Lower-triangular Controller. As expected there is no affect on q_2 and r_2, while the controller stabilizes q_1 as the steady state value of the response is zero.

IV. Robust Stability

Consider the descriptions given in Figure 9 and Figure 10 that shows uncertainty affecting the link from the controller to the plant, link from the plant to the controller and the link from the sub-controller K_1 to K_2 respectively (the uncertainty description is additive in Figure 9 and multiplicative in Figure 10 for convenience of exposition). The above uncertainty characterizations can be cast into the standard $M - \Delta$ framework of Figure 11 with $M \equiv M_1$ for the one channel case of Figure 9, and $M \equiv M_2$ for the two channel case of Figure 10 with M_1=

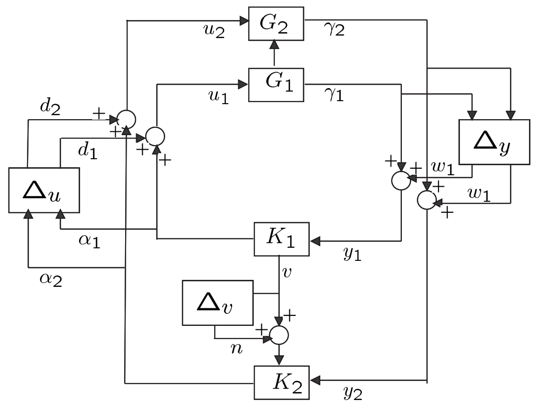

Fig. 9. Nested inter-connection with additive uncertainty in the inter-connections including the link between two controllers, based on the implementation of $K_1,\ K_2$ as described by the one single channel architecture of Figure 3

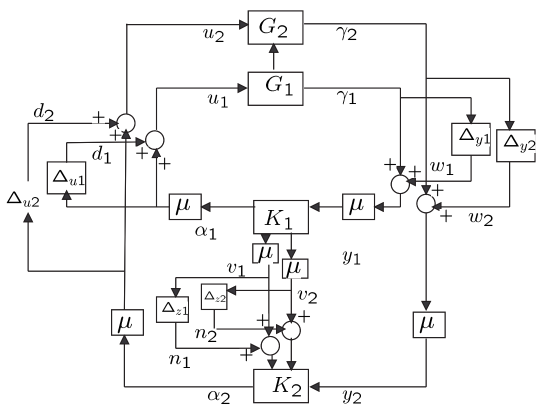

Fig. 10. Nested inter-connection with multiplicative uncertainty in the inter-connections including the two links between the two controllers, based on the implementation of $K_1,\ K_2$ as described by the two channel architecture of Figure 4

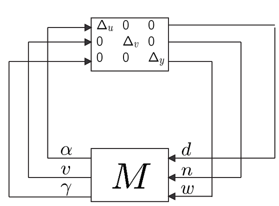

Fig. 11. The M-Δ configuration for single channel case

$$M_2 = \begin{pmatrix} \overline{Y}_r N_l & \begin{pmatrix} 0 \\ M_{22} \end{pmatrix} & \overline{Y}_r M_l \\ \begin{pmatrix} R\widetilde{N}_{11} & 0 \end{pmatrix} & 0 & \begin{pmatrix} R\widetilde{M}_{11} & 0 \end{pmatrix} \\ N_r \overline{X}_l & \begin{pmatrix} 0 \\ N_{22} \end{pmatrix} & \overline{X}_r M_l \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} \overline{Y}_r N_l & \begin{pmatrix} 0 & 0 \\ -M_{22}\widetilde{X}_{21} & M_{22}\widetilde{Y}_{21} \end{pmatrix} & \overline{Y}_r M_l \\ J\begin{pmatrix} \overline{Y}_r N_l \\ N_r \overline{X}_l \end{pmatrix} & 0 & J\begin{pmatrix} \overline{Y}_r M_l \\ \overline{X}_r M_l \end{pmatrix} \\ N_r \overline{X}_l & \begin{pmatrix} 0 & 0 \\ -N_{22}\widetilde{X}_{21} & N_{22}\widetilde{Y}_{21} \end{pmatrix} & \overline{X}_r M_l \end{pmatrix}$$

respectively where $R = \widetilde{Y}_{21} X_{11} - \widetilde{X}_{21} Y_{11}$, $J = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$. M_1 and M_2 are stable. It is to be noted that in the single channel case the dimension of Δ_v is smaller than the one in the two channel case. Thus the single channel implementation might prove attractive in cases where large number of nests are involved. However, it is evident that the M_1 does not depend affinely on the Youla parameter Q whereas M_2 is affine in Q. The non-affine nature of M_1 is due the term R that is not affine in Q. For the rest of the discussion we will focus on the two channel implementation described by Figure 10. We will consider the following class for uncertainty description: $\Delta_{LTV} = \{\Delta \in \mathcal{S}$ is linear time varying, $\|\Delta\|_{i\infty} < \infty\}$, where $\mathcal{S}$ characterizes the structure and subscript $i\infty$ stands for the ℓ_∞ and the ℓ_2 induced norm. Note that when $\mathcal{S}$ is given by the block diagonal structure $diag(\Delta_u, \Delta_v, \Delta_y)$ with Δ_u, Δ_y, and Δ_v being unstructured then the $M_2 - \Delta$ interconnection is robustly stable with respect to all $\Delta \in B\Delta_{LTV} := \{\Delta \in \Delta_{LTV} | \|\Delta\|_{i\infty} \le 1\}$ if and only if $\inf_{D\in\mathcal{D}} \|DM_2(Q)D^{-1}\|_{\ell_1} < 1$ where $\mathcal{D} = \{D = diag(1, d_1, d_2)$ with $d_i > 0\}$ [3]. Thus the problem for robust synthesis in this case reduces to the problem

$$\inf_{Q\in\ell_1} \inf_{D\in\mathcal{D}} \|DM_2(Q)D^{-1}\|_{\ell_1}.$$

This problem is nonconvex in the variables D and the Youla parameter Q. Recently in [4] a global solution to the above synthesis problem was achieved. Thus we have established an effective procedure to address the problem of synthesizing controllers for ℓ_1 robust synthesis when there is uncertainty in the nest to nest, subcontroller to subcontroller uncertainty when the uncertainty is described in the ℓ_∞ sense.

In the next interesting scenario, when the interaction channels send and receive information via Internet. Such single input single output channel can be characterized by channels that erase data in the form of loosing data-packet, using erasure channels.

Definition 4.1: ([1]): An Erasure channel with probability of erasure e, is a mapping $\mathcal{E} : \mathbb{R} \to \mathbb{R}$, $\mathcal{E} : w(k) \to \xi(k)w(k)$, where $\xi(k)$ is a discrete random variable with range space $\{0, 1\}$ and probability mass function given by $P_r(\xi(k) = 1) = (1 - e)$ and $P_r(\xi(k) = 0) = e$. $\xi(k)$'s are assumed to be independent and identically distributed.

Let, $w(k)$ be the sequence of data-packets transmitted, then $\xi(k)w(k)$ is the the sequence of packets received at the other end. Packet lost is same as no packet received. Also, let mean and variance of $\xi(k)$ be, $\mu := E\{\xi(k)\} = 1 - e$ and $\sigma^2 := E\{(\xi(k) - \mu)^2\} = e(1 - e)$. The QoS (Quality of Service) of erasure channel is defined in terms e by $QoS = 1 - e$.

Consider the two-channel architecture of the nested inter-connection as discussed in Section 2.2. (see Figure (4)). All the channels are assumed to be identical i.e. they have same QoS, which in turn means same rate of data-packet loss. The deterioration in QoS i.e. the loss of data-packets in this scenario can be characterized by using scalar multiplicative uncertainties of the form $\mu(1+\Delta)$ as shown in Figure (10).

In Figure (10), all channels have same μ, and Δ_{u1}, Δ_{u2}, Δ_{y1},Δ_{y2},Δ_{z1} and Δ_{z2} are independent identically distributed random variables, with zero mean and variance $\frac{\sigma^2}{\mu^2}$. Without loss of generality we will assume $\mu = 1$.

The Mean Square System for given $M_2 - \Delta$ inter-connection describes the evolution of the dynamics of $E\{x(k)x(k)^T\}$, where $x(k)$ is the state response of M_2. The $M_2 - \Delta$ inter-connection is Mean Square Stable[1], if for any initial state $x(0)$, $\lim_{k\to\infty} E\{x(k)x(k)^T\}$ = 0, where $x(k)$ is the state response of plant M_2, starting at $x(0)$. Note that M_2 is completely deterministic, while Δ is stochastic. It has been shown in [1] that the closed loop stability in Mean Square sense is equivalent to a robust control synthesis problem, with deterministic nominal model and stochastic structured uncertainty. Moreover, it is desirable to find the maximum rate of data-packet loss for which $M_2 - \Delta$ is Mean Square stable, i.e. maximizing e over all Mean Square Stable inter-connections, which is same as maximizing $\overline{\sigma}^2$ as $\overline{\sigma}^2$ = $\frac{e}{1-e}$.

Note that $G_{22}(\lambda = 0) = 0$ as G_{22} is strictly proper. Thus, we have $M_2(\lambda = 0)$ = $\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}$. Thus, the $M_2(\lambda = 0)$ is strictly upper triangular. Thus, the following lemma can be established based on [1]

Lemma 4.1: [1]. Since M_2 is stable, the following statements are equivalent:

- The $M_2 - \Delta$ inter-connection is Mean Square Stable.
- There exists a positive definite matrix P and a vector $\alpha \in \mathbb{R}^P$ of positive elements satisfying the following Linear Matrix Inequalities: $P > APA^T + \sum_{j=1}^{p} B_j \alpha_j B_j^T$
  $\alpha_i > \sigma^2 C_i P C_i^T + \sigma^2 \sum_{j=1}^{p} D_{ij}\alpha_j D_{ij}^T$, for i = 1,...,p.
- $\sigma^2 \inf_{\theta>0, Diag.} \|\theta M_2 \theta^{-1}\|_{MS}^2 < 1$
- $\rho(\sigma^2 \widehat{M_2}) < 1$ where $\rho(.)$ denoted the spectral radius and $\widehat{M_2} = \begin{pmatrix} \|M_{11}\|_2^2 & \cdots & \|M_{1p}\|_2^2 \\ \vdots & \ldots & \vdots \\ \|M_{p1}\|_2^2 & \cdots & \|M_{pp}\|_2^2 \end{pmatrix}$

Using Lemma 4.1, we can find the stablizing controller K by solving following synthesis problem [1]: $\tau^2 = \inf_{K\ stabilizing} \left\{ \inf_{\theta>0, Diag.} \|\theta M_2(K)\theta^{-1}\|_{MS} \right\}^2$ where $\|\cdot\|_{MS}$ is defined as $\| \ G \ \|_{MS} = \max_{i=1,\ldots,p} \sqrt{\sum_{j=1}^{p} \| G_{ij} \|_2^2}$ Again this problem can be effectively solved using the methodology presented in [4]. However note that the number of D scales has increased to five when compared to the two D scales in the ℓ_∞ setting. Similar result can be derived for generalized problem, i.e. n-nest problem, using $2(n-1)$ transmission channels.

V. Conclusions

In this paper two new architectures were provided that respect the nested nature of the controller and provide internal stability when sub-controller to sub-controller noise is included. In the second architecture where two transmission channels are used, the closed loop maps were parameterized by the Youla parameter Q in an affine manner. It was shown that uncertainty either modeled in a deterministic setting or a stochastic setting can be effectively addressed.

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