Phase Transitions on Fixed Connected Graphs and Random Graphs in the Presence of Noise

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***Abstract*— In this paper, we study phase transition behaviors emerging from the interactions between multiple agents in the presence of noise. We propose a simple discrete-time model in which a group of non-mobile agents form either a fixed connected graph or a random graph process, and each agent, taking bipolar value either $+1$ or -1, updates its value according to its previous value and the noisy measurements of the connected agents' values. We present proofs for the occurrence of the following phase transition behavior: At a noise level higher than some threshold, the system generates symmetric behavior; whereas at a noise level lower than the threshold, the system exhibits spontaneous symmetry breaking. We also verify the phase transition using simulations. This result may be found useful in the study of the collective behaviors of complex systems under communication constraints.**

I. Introduction

Phase transition in a system refers to the sudden change of a system property as some parameter of the system crosses a certain threshold value. Phase transitions have been observed in a wide variety of studies, such as in physics, chemistry, biology, complex systems, computer science, and random graphs, to list a few. It leads to long term attention in the literature, from physicists such as Ising [1] in the 1920's to mathematicians such as Erdos and Renyi [3] in the 1960's, from complex systems theorists such as Langton [7] in the 1990's to control engineers such as Olfati-Saber [11] in the 2000's.

Ising and other physicists have thoroughly studied the simple but "realistic enough" Ising model, for the understanding of phase transitions in magnetism, lattice gases, etc. In an Ising model, each node can have two values, and the neighboring nodes have an energetic preference to take the same value, under the constraints such as a temperature one. It is shown that, for an Ising model with dimension at least 2, a temperature higher than a critical point leads to symmetric behaviors (e.g., "melt" of magnetization, or vapor), whereas a temperature lower than that point leads to asymmetric behaviors (e.g., magnetization, or liquid). The Ising model is a discrete-time discrete-state model, and is closely related to the Hopfield networks and the cellular automata.

Erdos and Renyi [3] showed that, graphs of a size slightly less than a certain threshold are very unlikely to have some properties, whereas graphs with a few more edges are almost certain to have these properties. This is called phase transition of random graphs, see [5] for example.

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Viscek *et al* [14] showed that a two-dimensional nonlinear model exhibits a phase transition in the sense of spontaneous symmetry breaking as the noise level crosses a threshold. This model consists of a two-dimensional square-shaped box filled with particles represented as point objects in continuous motion. The following assumptions are also assumed: 1) the particles are randomly distributed over the box initially; 2) all particles have the same absolute value of velocity; and 3) the headings of the particles are randomly distributed. Each particle updates its heading using the average of its own heading and the headings of all other particles within a radius r, which is called the *nearest neighbor rule*. Included in this model is a random noise with a uniform probability distribution on the interval $[-\eta, \eta]$. The result of [14] is to demonstrate using simulations that a phase transition occurs when the noise level crosses a threshold which depends on the particle density. Below the threshold, all particles tend to align their headings along some direction, and above the threshold, the particles move towards different directions. Czirok *et al* [2] presented a one-dimensional model which also exhibits phase transition for a group of mobile particles. These two models are discrete-time continuous-state models.

Jadbabaie *et al* [4] provided a theoretical explanation for the noiseless scenario of the observed behavior in [14], with a slightly modified model. The model in [14] is a switched linear model (whereas the model in [2] is a switched non-linear model). Due to the noiseless assumption made in [4], the phase transitions observed in [14] will not occur. This model also assumes that over every finite period of time the particles are "jointly connected" for the length of the entire interval. Under these assumptions, Jadbabaie *et al* proved rigorously that the nearest neighbor rule leads to alignment of all particles. One might be interested in finding a Lyapunov function (preferably quadratic) to prove the convergence (alignment) property. However, Jadbabaie *et al* showed that a common quadratic Lyapunov function does not exist for the switched linear model. On the other hand, a non-quadratic Lyapunov function can be constructed to prove the convergence, as suggested by Megretski [8] and later independently found by Moreau [10].

Schweitzer *et al* [13] studied the spatial-temporal evolution of the population of two species in a square region, where the update scheme depends nonlinearly on the local frequency of species. In their stochastic model, the probability of inversion is non-zero even when all the neighbors are of opposite type. Depending on the probability of transition from one species to the other, the system evolves to either extinction of one species (agreement) or non-stationary co-existence or random co-existence (disagreement).

We refer to [11], [12] for a recent study of phase transitions (and the closely related consensus problems) over networks. Olfati-Saber [11] studied the consensus problem using a random rewiring algorithm [15] to connect nodes, and showed that the Laplacian spectrum of this network may undergos a dramatic shift, which is referred to as spectral phase transition and leads to extremely fast convergence to the consensus value.

In this paper, we propose a simple discrete-time discrete-state model in which a group of agents form either a fixed connected graph or a random graph process, and each agent (node) updates its value according to the noisy measurements of the neighboring agent values. We prove that, when the noise level crosses some threshold from above, the system exhibits spontaneous symmetry breaking. We may view that the high noise level corresponds to high temperature (or strong thermal agitation), where the molecules exhibit disorder and symmetry; and the low noise level corresponds to low temperature, where the molecules exhibit order and asymmetry.

We may also interpret our phase transition in the consensus problem scenario, where the disagreement due to unreliable communication is replaced by agreement when the communication quality improves to a certain level. However, unlike the average-consensus problem (cf. [12]) with the properties that, 1) there exist invariant quantities during the evolutions, and 2) the limiting behaviors reach the average of the initial states of the systems, we reach a consensus without these properties when the noise level is low. This is because the presence of noises prevents the conserving of sum of the agents' values during the evolution. We conjecture that there may exist an invariant quantity based on the entropy flows of the system, see [9] for the study of entropy flows. We remark that a more thorough study of the consensus problem raised in this paper is beyond the scope of this paper and will be pursued elsewhere.

We demonstrate how phase transition can rise in rather simple models. The phase transition in a fixed connected graph presented in this paper is simpler than the phase transition in the Ising model (for example, the Ising model needs dimension 2 to generate phase transition, whereas our phase transition occurs for any dimension). To the best of our knowledge, this model is one of the simplest that exhibits phase transition in a fixed graph, and is mathematically provable to generate phase transition. Additionally, the phase transition on a random graph is also simpler than the phase transition on a random graph observed in [3]. Compared with the models in [14] and [2], our models have discrete-states and do not allow the mobility of agents, which greatly simplifies the analysis of the phase transition behaviors.

The simplicity of our phase transitions may help us to identify the essence of general phase transition phenomena. Our study also sheds light to the research on the consensus problems, cooperation of multiple-agent systems, and collective behaviors of complex systems, all under communication constraints.

Organization: In Section 2 we introduce the models. In Section 3 we state our main results and provide the proofs. In Section 4 we present numerical examples. Finally we conclude the paper and discuss future research directions.

II. Models on the graphs

This section introduces some of the terms that are frequently used in this paper as well as the two models to be investigated. We focus only on undirected graphs.

A. Graphs and random graph processes

A *graph* $G := (V, E)$ consists of a set $V := \{1, 2, ..., N\}$ of elements called vertices or nodes, and a set E of node pairs called edges, with $E \subseteq E_c := \{(i, j)|i, j \in V\}$. Such a graph is *simple* if it has no self loops, i.e. $(i, j) \notin E$ if $i = j$. We consider simple graphs only. A graph G is *connected* if it has a path between each pair of distinct nodes i and j, where by a *path* between nodes i and j we mean a sequence of distinct edges of G of the form $(i, k_1), (k_1, k_2), \ldots, (k_m, j) \in E$. Radius r from node i to node j means that the path length, i.e., the number of edges connecting i to j, is equal to r.

A *fixed* graph G has a node set V and an edge set E that consists of deterministic edges, that is, the elements of E are deterministic and do not change dynamically with time.

A *random* graph G consists of a node set V and an edge set $E := E(\omega)$, where $\omega \in \Omega$, $(\Omega, \mathcal{F}, P)$ forms a probability space. Here Ω is the set of all possible graphs (with total number $n := 2^{N(N-1)/2}$), $\mathcal{F}$ is the power set of Ω, and P is a probability measure that assigns a probability to every $\omega \in \Omega$. In this paper, we focus on the well-known Erdos random graphs only [5], namely, it holds that

$$P(\omega) = \frac{1}{n}. \tag{1}$$

In other words, we can view that each $E(\omega)$ is the result of $N(N-1)/2$ independent tosses of fair coins, where a head corresponds to switching on the associated edge. Notice that the introduction of randomness to a graph implies that, all results in random graph theory hold asymptotically and in a probability sense, such as "hold with probability one".

A *random graph process* is a stochastic process that describes a random graph evolving with time. In other words, it is a sequence $\{G(k)\}_{k=0}^{\infty}$ of random graphs (defined on a common probability space $(\Omega, \mathcal{F}, P)$) where k is interpreted as time [5]. Thus, for a random graph process, the edge set changes with k, and we denote it at time k as $E(k)$. In this paper, we assume that the edge formation at time k is independent of that at time l, if $k \neq l$.

The *neighborhood* $N_i(k)$ of the ith node at time k is a set consisting of all nodes within radius 1, including the ith node itself. The value that a node assumes is its *node value*. The *valence* or *degree* of the ith node is $(|N_i(k)| - 1)$, where $|N_i(k)|$ denotes the number of elements in $N_i(k)$. The *adjacency matrix* of $G(k)$ is an $N \times N$ matrix whose (i, j)th entry is 1 if the node pair $(i, j) \in E(k)$ and 0 otherwise.

B. System on a graph

A *system on a graph* consists of a graph, fixed or forming a random process, an initial condition that assigns each node a node value, and an update rule of the node values. In this paper, we assume that each node can take value either $+1$ or -1, and the *update rule* for the ith node at the $(k+1)$th instant is given by

$$x_i(k+1) = \text{sign}\left(v_i(k) + \xi_i(k)\right), \tag{2}$$

where $\xi_i(k)$ is the *noise* random variable, uniformly distributed in interval $[-\eta, \eta]$ and independent across time and space, and

$$v_i(k) := \frac{\sum_{j \in N_i(k)} x_j(k)}{|N_i(k)|}; \tag{3}$$

that is, $v_i(k)$ is the average of the node values in the neighborhood $N_i(k)$. Here η is called the *noise level*. This update rule resembles the one in [2], with their antisymmetric function being replaced by a sign function. It may also be viewed as a specific update rule for a Hopfield neuron with noisy connections to others.

The *state of the system* at time instant k, denoted $S(k)$, is the sum of the current values of all the nodes in the graph. We call a state *transient* if this state reappears with probability strictly less than one. We call a state *recurrent* if this state reappears with probability one. We call a state S *absorbing* if the one-step transition probability from S to S is 1.

C. Model with a fixed graph

The first model considered is a system on a fixed graph. In this model, the node connections or the edges remain unchanged throughout. Hence, every node has a fixed neighborhood at all time, and the degree of each node as well as the adjacency matrix are constant. The node value gets updated according to the update rule (2). We will assume that the fixed graph is connected. An example of such a fixed graph model is a communication network with fixed nodes and fixed but noisy channels. Another example is a Hopfield network with fixed neurons and fixed but noisy connections.

D. Model with a random graph process

The second model considered is a system on a graph forming a random process. In this model, the node connections, namely the edges of the random graph, change dynamically throughout, and the edge formations at time k are random according to distribution $P(k)$. Hence every node may have different neighborhoods at different times, and the adjacency matrix and degrees change with time. The node value gets updated also according to the update rule (2). An example of this model could be an ad-hoc sensor network in which the communication links between the sensors are changing. Another example is an erasure network in which the communication channels are noisy and erasing with some probability, see for example [6].

In both models, the state of the system takes values in the set $\mathcal{N} := \{-N, -N+2, \cdots, N-2, N\}$, where $N \geq 2$ is the total number of nodes (agents). Note that $|\mathcal{N}| = N+1 \geq 3$. Both models also form a Markov chain, since the next state does not depend on previous state if the current state is given.

We use $\xi(k)$ to represent $(\xi_1(k), \cdots, \xi_N(k))$, ξ^k to represent $(\xi(0), \cdots, \xi(k))$, and G^k to represent $(G(0), \cdots, G(k))$.

III. Main results and proofs

Our main result states that, for a system on a fixed connected graph or on a graph forming a random process, there is a provable phase transition when the noise level crosses some threshold. Here phase transition is used in the sense that the symmetry exhibited at high noise level is broken suddenly when the noise level crosses the threshold from above, or equivalently the disagreement of the nodes at high noise level becomes agreement below the threshold. In what follows, we first discuss the case in which the graph has a fixed structure, and then the case in which the graph forms random process.

A. Model with a fixed graph

Proposition 1. *For any given fixed connected graph, let D be the maximum number of nodes in one neighborhood.*

i) If the noise level is such that $\eta \in (1 - 2/D, 1]$, then all nodes will converge to agreement, namely all nodes will eventually converge to either all $+1$s or all -1s.

ii) If the noise level is such that $\eta > 1$, then $\mathbf{E}_{S(0),\xi^{k-1}} S(k)$ tends to zero as k goes to infinity, i.e., all nodes will converge to disagreement in which approximately half of the nodes are $+1$s and the other half are -1s at each time.

Remark 1. Notice that $(1 - 2/D)$ is guaranteed to be nonnegative for any connected graph with more than one node, since $D \geq 2$. Note also that if $\eta < 1 - 2/D$, the system may converge to states not equal to $\pm N$. To see this, simply consider a one-dimensional cellular automaton with N nodes forming a circle. The neighborhood of a node is defined as one node to the left, one node to the right, and itself. Therefore $D = 3$, and if $\eta < 1/3$, the update rule becomes a majority voting rule. Then the initial state $S(0)$ of the system with alternate $+1$s and -1s will leads to constant oscillations between $S(0)$ and a left shift of $S(0)$, i.e., it will not reach agreement if $\eta < 1/3$. However, this does not mean that in general our condition $1 \geq \eta > 1 - 2/D$ is a necessary condition for agreement; a sufficient and necessary condition is under current investigation. Attractors like this $S(0)$ may be viewed as local attractors (whereas $\pm N$ may be viewed as global attractors) but they can be removed by considering a *randomized* graph, which we will study in the next subsection.

The proof of Proposition 1 needs the following lemmas.

Lemma 1. *For any given fixed connected graph, if $\eta \in (1 - 2/D, 1]$, then the states $\pm N$ are absorbing, and all other states are transient.*

Lemma 2. *For any given fixed connected graph, if $\eta > 1$, then the states form an ergodic Markov chain with a unique steady-state distribution for any initial condition.*

Proof of Lemma 1: Notice that at state $+N$, the noise is not strong enough to flip any node value. So $\pm N$ are absorbing. On the other hand, all other states are neither absorbing nor recurrent. To see this, let $M \neq \pm N$ be any state. In this case, M contains a mixture of $+1$s and -1s. For M, we can always find a node with value $x_i(k) = -1$ whose neighborhood $N_i(k)$ (including $x_i(k)$ itself) contains both $+1$s and -1s. If this were not true, then for any node with value -1, its neighbors must be all -1s. By induction, any nodes within any radius from a node with value -1 are -1s. However, this contradicts the fact, due to connectivity of the graph, that there must exist a path between an arbitrary -1 and an arbitrary $+1$. Thus such $x_i(k)$ and $N_i(k)$ containing both $+1$s and -1s exist.

Then for such $x_i(k)$, it holds that

$$|v_i(k)| \le \frac{D-2}{D}, \tag{4}$$

with equality if only one node in $N_i(k)$ has a different sign than all other nodes and if $N_i(k)$ contains D nodes. Hence a noise larger than $(D-2)/D$ flips $x_i(k)$. Precisely,

$$\begin{aligned} & \Pr(x_i(k+1) = +1 | x_i(k) = -1) \\ = \;& \Pr\left(v_i(k) + \xi_i(k) > 0\right) \\ \ge \;& \Pr\left(\xi_i(k) > \frac{D-2}{D}\right) \\ = \;& \frac{1}{2}\left(1 - \frac{D-2}{D\eta}\right) > 0. \end{aligned} \tag{5}$$

Thus, for state M, the probability that only s_i flips and no other node changes its value is non-zero. This follows that, with a positive probability M will increase to $M+2$. Similarly, with a positive probability M can decrease to $M-2$. Since $M \neq \pm N$ is an arbitrary state, by induction, the probability of transition (in possibly multiple steps) from M to $\pm N$ is nonzero. So M is transient. ■

Proof of Lemma 2: It is sufficient to prove that the state forms an irreducible and aperiodic Markov chain.

To see the irreducibility, note that if $\eta > 1$, $M \neq \pm N$ is transient, similar to Lemma 1. Additionally, $\pm N$ are not absorbing. For state $+N$, it holds that

$$\begin{aligned} & \Pr(x_i(k+1) = -1 | x_i(k) = +1) \\ = \;& \Pr\left(\tfrac{\sum_{j\in N_i(k)} +1}{|N_i(k)|} + \xi_i(k) < 0\right) \\ = \;& \Pr(\xi_i(k) < -1) \\ = \;& \eta - 1 > 0, \end{aligned} \tag{6}$$

so state $+N$ can jump to $N-2$ with a positive probability. Similarly $-N$ can also jump to $-N+2$ with a positive probability. Then this Markov chain is irreducible.

To see the aperiodicity, note that any state M can increase to $M+4$ or decrease to $M-4$ with a positive probability, if $M+4 \le N$ or $M-4 \ge -N$. Therefore, the loop $M \to M+2 \to M-2 \to M$ has period 3, but the loop $M \to M+2 \to M$ has period 2. Then M is aperiodic. ■

Proof of Proposition 1: If $\eta \in (1-2/D, 1]$, from Lemma 1, the associated Markov chain will converge to either $+N$ or $-N$ with probability 1, namely agreement. If $\eta > 1$, from Lemma 2 we know that the associated Markov chain is ergodic, and notice that the Markov chain has a symmetric structure for state $+M$ and $-M$, $M = N, N-2, \cdots, -N+2, -N$. Then

$$\pi(+M) = \pi(-M), \tag{7}$$

where $\pi(+M)$ is the stationary probability of state $+M$. Hence the expectation of state

$$\mathbf{E}_\pi S := \sum_{M\in\mathcal{N}} \pi(M)M = 0. \tag{8}$$

Therefore, $\mathbf{E}S(k)$ converges to zero, and asymptotically the number of $+1$s and -1s will become roughly equal. ■

B. Model with a random graph process

For an Erdos random graph, we assume that the edge connections are randomly and independently changing from time to time. The randomization of the connections symmetrizes the behavior of each node, and helps us to achieve agreement for an arbitrarily small but positive noise level.

Proposition 2. *Consider an Erdos random graph process.*

i) If the noise level is such that $0 < \eta \le 1$, then all nodes will converge to agreement, namely the state will converge to $+N$ or all $-N$.

ii) If the noise level is such that $\eta > 1$, then $\mathbf{E}_{S(0),\xi,G}S(k)$ exponentially converges to zero as k goes to infinity, i.e., the system will exponentially converge to disagreement in which about half of the node values are $+1$s and the other half are -1s.

Note that $S(k)$ contains randomness of 1) the initial condition $S(0)$, 2) the noises ξ^{k-1}, and 3) the graphs G^{k-1}. Then we write the expectation of $S(k)$ over all randomness as $\mathbf{E}_{S(0),\xi^{k-1},G^{k-1}}S(k)$. The proof of this proposition needs the following lemmas.

Lemma 3. *For any Erdos random graph process, if $0 < \eta \le 1$, then $\pm N$ are absorbing, and all other states are transient.*

Lemma 4. *For any Erdos random graph process, if $\eta > 1$, then it holds that $\mathbf{E}_{S(0),\xi^{k-1},G^{k-1}}S(k)$ exponentially tends to zero as k goes to infinity from any initial state.*

Proof of Lemma 3: For any Erdos random graph process, if $0 < \eta \le 1$, it is easy to see that $\pm N$ are absorbing. For any other state $M \neq \pm N$, it holds that M must be a mixture of both $+1$s and -1s. Hence we can find in M that $x_i(k) = -1$ and $x_j(k) = +1$. By the randomness of the graph edges, the probability that x_i is connected to x_j only is positive, since each one of the n graphs has a positive probability. This follows that $v_i(k) = 0$ and hence arbitrarily small but positive noise may flip x_i with a positive probability. In addition, all nodes other than x_i have a positive probability to keep their previous values, thus with a positive probability M can increase to $M+2$. Similarly, with a positive probability M can decrease to $M-2$. Thus any states other than $\pm N$ are transient. ■

Proof of Lemma 4: For any Erdos random graph, if $\eta > 1$, then no state is an absorbing one, since with a positive probability the noise can flip any node value in any configuration. Therefore, with a nonzero probability the state of the system can jump to any other states.

Now let us analyze the evolution of $\mathbf{E}S(k)$. Fix the time k. Assume $x_i(k)$ is given for each i. Then $x_i(k+1)$ is given by (2). The randomness in $x_i(k+1)$ is due to the noise $\xi_i(k)$ and the graph $G(k)$. It holds that

$$\begin{aligned}
& \mathbf{E}_{\xi_i(k),G(k)} x_i(k+1) \\
= \ & \mathbf{E}_{\xi_i(k),G(k)} \text{sign}(v_i(k)+\xi_i(k)) \\
= \ & \Pr(v_i(k)+\xi_i(k)>0)\times(+1) \\
& +\Pr(v_i(k)+\xi_i(k)<0)\times(-1) \\
= \ & \Pr(\xi_i(k)>-v_i(k)) - \Pr(\xi_i(k)<-v_i(k)) \\
= \ & \sum_{v_i(k)} \Pr(\xi_i(k)>-v_i(k)|v_i(k))\Pr(v_i(k)) \\
& -\sum_{v_i(k)} \Pr(\xi_i(k)<-v_i(k)|v_i(k))\Pr(v_i(k)) \\
= \ & \sum_{v_i(k)} [\Pr(\xi_i(k)>-v_i(k)|v_i(k)) \\
& -\Pr(\xi_i(k)<-v_i(k)|v_i(k))]\Pr(v_i(k)) \\
= \ & \sum_{v_i(k)} \left[\frac{\eta+v_i(k)}{2\eta} - \frac{\eta-v_i(k)}{2\eta}\right]\Pr(v_i(k)) \\
= \ & \sum_{v_i(k)} \frac{v_i(k)}{\eta}\Pr(v_i(k)) \\
= \ & \frac{1}{\eta}\mathbf{E}_{G(k)} v_i(k).
\end{aligned} \tag{9}$$

Note that the above computation is conditioned on $x_i(k)$, but we omit the conditioning on $x_i(k)$ to simplify notation.

Then we compute $\mathbf{E}_{G(k)} v_i(k)$, the expectation of the average of node values in a neighborhood, w.r.t. all possible n graph structures. Let us count in the n graph structures the number of different neighborhoods containing $x_i(k)$. Among those neighborhoods containing $x_i(k)$, there are

$$2^{(N-1)(N-2)/2} \times \binom{N-1}{m} \tag{10}$$

types of neighborhoods with $(m+1)$ node inside, $m = 0, 1, \cdots, N-1$. Therefore,

$$\begin{aligned}
& \mathbf{E}_{G(k)} v_i(k) \\
= \ & \sum_r r\Pr(v_i(k)=r) \\
= \ & \sum_r \frac{x_i(k)}{|N_i(k)|}\Pr(v_i(k)=r) + \\
& \sum_r \frac{\sum_{j\in N_i(k), j\neq i} x_j(k)}{|N_i(k)|}\Pr(v_i(k)=r) \\
= \ & c_1 x_i(k) + \sum_{j\in N_i(k), j\neq i} c_2 x_j(k),
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
c_1 &:= \frac{2^{(N-1)(N-2)/2}}{n}\sum_{m=0}^{N-1}\binom{N-1}{m}\times\frac{1}{m+1} \\
c_2 &:= \frac{2^{(N-1)(N-2)/2}}{n(N-1)}\sum_{m=1}^{N-1}\binom{N-1}{m}\times\frac{m}{m+1}.
\end{aligned} \tag{12}$$

This yields

$$\mathbf{E}_{\xi_i(k),G(k)} x_i(k+1) = \frac{1}{\eta}\left(c_1 x_i(k) + c_2\sum_{j\neq i} x_j(k)\right), \tag{13}$$

and hence

$$\begin{aligned}
& \mathbf{E}_{\xi(k),G(k)} S(k+1) \\
= \ & \sum_{i=1}^{N} \mathbf{E}_{\xi_i(k),G(k)} x_i(k+1) \\
= \ & \frac{1}{\eta}\left(c_1 S(k) + c_2(N-1)S(k)\right) \\
= \ & \frac{1}{\eta}S(k)\frac{1}{2^{N-1}}\left(\sum_{m=0}^{N-1}\binom{N-1}{m}\right) = \frac{1}{\eta}S(k).
\end{aligned} \tag{14}$$

Since

$$\begin{aligned}
& \mathbf{E}_{S(0),\xi^k,G^k} S(k+1) \\
= \ & \mathbf{E}_{S(0),\xi^{k-1},G^{k-1}}[\mathbf{E}_{\xi(k),G(k)} S(k+1)] \\
= \ & \mathbf{E}_{S(0),\xi^{k-1},G^{k-1}}[\frac{1}{\eta}S(k)].
\end{aligned} \tag{15}$$

To summarize, we have

$$\mathbf{E}S(k+1) = \frac{1}{\eta}\mathbf{E}S(k). \tag{16}$$

Since $\eta > 1$, the above recursion converges to zero exponentially. ∎

Proof of Proposition 2: If $0 < \eta \leq 1$, from Lemma 3, the system state will converge to the absorbing states eventually with probability 1, namely agreement. If $\eta > 1$, from Lemma 4, the system state will converge to zero exponentially with probability 1. ∎

Remark 2. The proof of Lemma 3 is based on a rather conservative argument only, but it utilizes the randomness of graph structure and leads to the result easily. The proof of Lemma 4 relies on the idea that the randomness of the edge formation symmetrizes the behavior of each node. It is straightforward to generalize the proof to a binomial random graph, where the probability of forming an edge is changed from 0.5 to an arbitrary $p \in (0,1)$. We also see that the decay exponent of the mean state of the system is

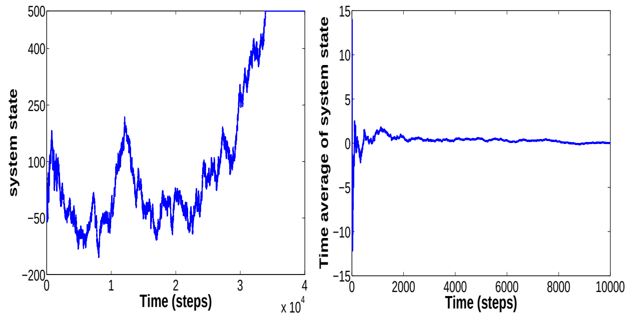
$$-\frac{1}{k}\log\frac{\mathbf{E}S(k)}{\mathbf{E}S(0)} = -\log\frac{1}{\eta} = \log\eta, \tag{17}$$

and the larger the noise level is, the faster the convergence.

IV. Numerical results

A. Fixed graph case

For the fixed graph case, we consider a fixed one-dimensional 500-agent system to illustrate the phase-transition phenomenon. The agents are listed along a circle and each agent has two neighbors. The initial value of every agent is randomly assigned to be $+1$ or -1 according to an arbitrary distribution. The simulation results are given in Figure 1 (a) and (b). In Figure 1 (a), the vertical axis represents the state of the system, and the horizontal axis represents the simulation steps. Figure 1 (a) shows that, if the noise level is such that $1/3 < \eta \leq 1$, then all node values converge to agreement of all $+1$s or all -1s, that is, the state of the system is +500 or -500. In Figure 1 (b), the vertical axis represents the time average of the state, and the horizontal axis is for the simulation steps. By ergodicity of the system, the time average should converge to the ensemble average of the state. Figure 1 (b) shows that, if the noise level is such that $\eta > 1$, then all node values converge to disagreement in which about half of the node values are $+1$s and the other half are -1s. Clearly, the noise level equal to 1 is the critical level of the phase transition.

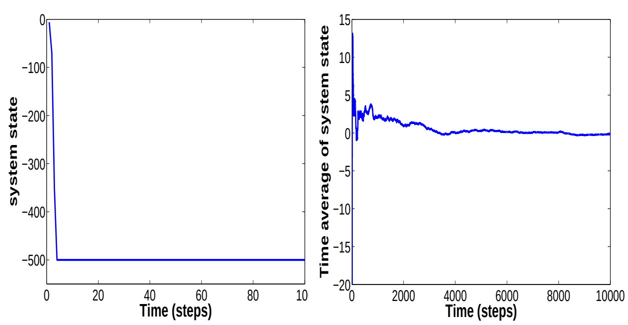


(a) (b)

Fig. 1. Fixed graph simulation. (a) Noise level is 0.75, and the state converges to agreement of all $+1$s. (b) Noise level is 2, and the state converges to disagreement in which about half of the states are $+1$s and the other half are -1s.

B. Random graph process case

For the random graph process case, in our simulation we consider *binary random graphs*. In a binary random graph, each edge has a probability p to be formed at each time step and is independent of all other edges and other times. This means that to generate such a binary random graph, we only need to generate at each step an adjacency matrix whose entries in the upper triangular part are independent and identically distributed. The initial value of every agent is randomly assigned to be $+1$ or -1 according to an arbitrary distribution. The simulation results are shown in Figure 2 (a) and (b), and are similar to the fixed connected graph case, except that in the random graph case, an arbitrarily small but positive noise level can lead to agreement.



(a) (b)

Fig. 2. (a) System is randomly connected, noise level is 0.005, $p = 0.1$, and all states converge to agreement of all -1s. (b) System is randomly connected, noise level is 2, $p = 0.2$, and the state converges to disagreement in which about half of the states are $+1$s and the other half are -1s.

We can also compute the decay exponent of $\mathbf{E}S(k)$ from the numerical results. Note that to obtain the probability mean $\mathbf{E}S(k)$ numerically, we can run the random process many times and take the average of the states. See Figure 3 for the simulated decay exponent and the theoretic decay exponent $\log_2 \eta$, which are almost identical.

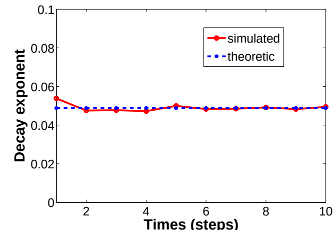


Fig. 3. The simulated decay exponent (averaged over 10,000 independent trials and $\eta = 1.05$) and the theoretic decay exponent.

V. CONCLUSIONS AND FUTURE WORK

We have shown that for fixed connected graphs, if the noise level is greater than $1 - 2/D$ and less than 1, all the agents reach an agreement, i.e. the state of the system converges to $\pm N$, the only absorbing states of the system. For noise level larger than 1, the group of agents fails to reach any agreement. Thus, phase transition occurs at $\eta = 1$. For random graph processes, we have shown that the system reaches agreement even for noise level smaller than $1-2/D$. This is because randomization is immune to the artifacts (or unwanted local attractors) for smaller noise which stops fixed graph from reaching any agreement. But, the tradeoff is that in randomization, the nodes' neighbors are random, and therefore the neighbors may not be "geographically close", which might not be feasible in practical situations. Our study was concentrated on the leaderless case. The leader case is when there is a leader with a fixed value, and will try to convince all other agents to follow him. Simulation obtained in this case suggested that a complete analysis is a bit involved especially in the high noise regime, which is subject to further research. Another direction could be to obtain a suitable Lyapunov function for the models. One advantage of doing so is that the Lyapunov function based approach may be extended to rather general nonlinear systems, as suggested by [8], [10]. The Lyapunov function is preferably a quadratic one, leading to stability in the mean-square sense, which is stronger than the stability in the mean sense obtained in this paper. The applications of our approach and results are also subject to future research, including the extension of our approach to more realistic models; note that our models in this paper are rather simple and not realistic enough, though the simplicity helped us to completely characterize the phase transition and may help us to identify the essence of phase transitions. We will also explore the connections of our model to relevant models, such as the Ising models, Hopfield networks, cellular automata, other random graphs, etc.

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